CALT-68-2499
PUPT-2116
hep-th/0405153

Higher Impurity AdS/CFT Correspondence in the Near-BMN Limit

Curtis G. Callan, Jr.[a], Tristan McLoughlin[b], Ian Swanson[b]

[a] Joseph Henry Laboratories
Princeton University
Princeton, New Jersey 08544, USA

[b] California Institute of Technology
Pasadena, CA 91125, USA

Abstract

The pp-wave/BMN limit of the AdS/CFT correspondence has exposed the Maldacena conjecture to a new regimen of direct tests. In one line of pursuit, finite-radius curvature corrections to the Penrose limit (which appear in inverse powers of the string angular momentum J) have been found to induce a complicated system of interaction perturbations to string theory on the pp-wave; these have been successfully matched to corresponding corrections to the BMN dimensions of $\mathcal{N} = 4$ super Yang-Mills (SYM) operators to two loops in the 't Hooft coupling λ. This result is tempered by a well-established breakdown in the correspondence at three loops. Notwithstanding the third-order mismatch, we proceed with this line of investigation by subjecting the string and gauge theories to new and significantly more rigorous tests. Specifically, we extend our earlier results at $O(1/J)$ in the curvature expansion to include string states and SYM operators with three worldsheet or R-charge impurities. In accordance with the two-impurity problem, we find a perfect and intricate agreement between both sides of the correspondence to two-loop order in λ and, once again, the string and gauge theory predictions fail to agree at third order.

February 1, 2008

1 Introduction

Recent explorations of the AdS/CFT correspondence have focused on comparing the spectrum of the type IIB superstring boosted to a near-lightcone (large angular momentum J) trajectory on $AdS_5 \times S^5$ with the anomalous dimensions of large R-charge operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. This line of investigation was launched on the string side by Metsaev [1, 2], who showed that the string worldsheet dynamics can be reduced to a system of free massive fields by taking a Penrose limit of the target geometry that reduces $AdS_5 \times S^5$ to a pp-wave. On the gauge theory side, Berenstein, Maldacena and Nastase [3] identified the relevant class of gauge theory operators (commonly referred to as BMN operators) whose dimensions can be calculated perturbatively and matched to the string theory energy spectrum on the pp-wave.

Attempts to push the original results further have gone in two independent directions. In the gauge theory, the calculation of anomalous dimensions of BMN operators has been greatly simplified by Minahan and Zarembo's discovery that the problem can be mapped to that of computing the energies of certain integrable spin chains [4]. Based on this development, calculations in certain sectors of the theory have been carried out to three loops in the 't Hooft coupling λ [5, 6].[1] At the same time, the quantization of the Green-Schwarz string in the $AdS_5 \times S^5$ background has developed far enough to enable perturbative computations of the effect of worldsheet interactions on the spectrum of the string when it is boosted to large, but finite, angular momentum J [7, 8, 9]. These two approaches lead to different expansions of operator anomalous dimensions (or string eigenenergies): on the gauge theory side, one naturally has an expansion in the coupling constant λ which is typically exact in R-charge; on the string theory side one has an expansion in inverse powers of angular momentum J (the dual of gauge theory R-charge) which is exact in λ.

The expansion on the string side is difficult and has so far been carried out to $O(1/J)$ for 'two-impurity' states (i.e. states with two string oscillators excited). The resulting functions of the loop expansion parameter λ can be compared with the large R-charge expansion of two-impurity BMN operators in the gauge theory to provide new and stringent tests of the AdS/CFT correspondence. As mentioned above, recently developed gauge theory technology has made it possible to compute anomalous dimensions of certain two-impurity BMN operators out to three-loop order. The agreement between dual quantities is perfect out to two-loop order but, surprisingly, seems to break down at three loops [8, 9]. Exactly what this means for the AdS/CFT correspondence is not yet clear but, given the circumstances, it seems appropriate to at least look for further data on the disagreement in the hope of finding some instructive systematics. The subject of this paper is to pursue one possible line of attack in which we extend the calculations described above to higher-impurity string states and gauge theory operators. The extension of our two-impurity results to higher impurities is not a straightforward matter on either side of the correspondence and gets more complex as the number of impurities increases. We focus here on the three-impurity case, where we

[1] We note that the conjectural three-loop computation of [5] was solidified by field theoretic methods in [6].

obtain results which validate our methods for quantizing the Green-Schwarz superstring; the agreement with gauge theory at one and two loops is impressive, though we also confirm the previously observed breakdown of agreement at three-loop order.

In section 2, we give a brief review of string quantization on $AdS_5 \times S^5$. Since we use exactly the same methods and notation as in our recent treatment of the two-impurity problem, we can refer the reader to our long paper on that topic for a more detailed exposition of our methods [9]. In sections 3 and 4 we present the details of the diagonalization of the perturbing string worldsheet Hamiltonian on degenerate subspaces of three-impurity states. We give a compressed discussion of general strategy, concentrating on the aspects of the problem which are new to the three-impurity case. An interesting new element is that the non-interacting degenerate subspace breaks up into several different supersymmetry multiplets so that the detailed accounting of multiplicities and irrep decomposition amounts to a stringent test that the quantization has maintained the correct nonlinearly realized superconformal symmetries. Section 5 is devoted to the comparison of the string theory spectrum with gauge theory anomalous dimensions. Since gauge theory results have not been obtained elsewhere for higher-impurity operators to the order we need, we generate our own data by doing numerical analysis, along the lines of [10], of the various higher-loop spin chains onto which the gauge theory anomalous dimension problem has been mapped. This is an interesting problem in its own right and we give only a brief description of our methods, referring the reader to a separate methods publication [11] for details. We are able to take scaling limits of the numerical spin chain analyses that allow us to make clean contact with the string theory results. We find perfect agreement through two-loop order and, once again, breakdown at three loops. Section 6 is devoted to discussion and conclusions.

Overall, the three-impurity regime of the string theory offers a much more stringent test of the duality away from the full plane-wave limit. While we are unable to offer a solution to the disagreement with gauge theory at three loops, we can confirm that that the complicated interacting worldsheet theory at $O(1/\widehat{R}^2)$ in the curvature expansion is properly quantized and correct to two loops in λ.

2 String quantization on $AdS_5 \times S^5$: brief review

A standard presentation of the $AdS_5 \times S^5$ metric is

$$ds^2 = \widehat{R}^2(-\cosh^2\rho\ dt^2 + d\rho^2 + \sinh^2\rho\ d\Omega_3^2 + \cos^2\theta\ d\phi^2 + d\theta^2 + \sin^2\theta\ d\tilde{\Omega}_3^2)\ , \qquad (2.1)$$

where $\widehat{R}$ is the radius of each subspace, and $d\Omega^2$, $d\tilde{\Omega}_3^2$ are separate three-spheres. The spacetime radius is related to the coupling of the dual $SU(N_c)$ gauge theory by $\widehat{R}^4 = \lambda(\alpha')^2 = g_{YM}^2 N_c (\alpha')^2$. The reparameterization

$$\cosh\rho = \frac{1+z^2/4}{1-z^2/4} \qquad \cos\theta = \frac{1-y^2/4}{1+y^2/4} \qquad (2.2)$$

simplifies the spin connection and is particularly convenient for constructing the superstring action in the coset formalism [8, 9]. In these coordinates, the target-space metric takes the

form

$$ds^2 = \widehat{R}^2\left[-\left(\frac{1+\frac{1}{4}z^2}{1-\frac{1}{4}z^2}\right)^2 dt^2 + \left(\frac{1-\frac{1}{4}y^2}{1+\frac{1}{4}y^2}\right)^2 d\phi^2 + \frac{dz_k dz_k}{(1-\frac{1}{4}z^2)^2} + \frac{dy_{k'}dy_{k'}}{(1+\frac{1}{4}y^2)^2}\right] . \qquad (2.3)$$

The transverse $SO(8)$ breaks into $SO(4) \times SO(4)$, which is spanned by $z^2 = z_k z^k$ with $k = 1, \ldots, 4$, and $y^2 = y_{k'}y^{k'}$ with $k' = 5, \ldots, 8$. We use lightcone coordinates

$$t = x^+ , \qquad \phi = x^+ + x^-/\widehat{R}^2 , \qquad (2.4)$$

leading to conjugate lightcone momenta

$$-p_+ = \Delta - J , \qquad -p_- = i\partial_{x^-} = \frac{i}{\widehat{R}^2}\partial_\phi = -\frac{J}{\widehat{R}^2} , \qquad (2.5)$$

where J is the conserved angular momentum conjugate to ϕ and Δ is the conserved energy conjugate to t (which will eventually be identified with gauge theory operator dimensions). The technical benefits of this lightcone coordinate choice are explained in [8, 9]). When quantizing the string in the BMN limit, p_- is held fixed while J (the angular momentum of the string) and $\widehat{R}$ (the scale of the geometry) are both taken to be large. This is possible because $p_-\widehat{R}^2 = J$ (2.5) and we can regard this as either a large-J or large-$\widehat{R}$ limit; we will pass freely between the two.

The large-$\widehat{R}$ expansion of (2.3) is

$$\begin{aligned} ds^2 \;\approx\; & 2dx^+dx^- - (x^A)^2(dx^+)^2 + (dx^A)^2 \\ & + \frac{1}{\widehat{R}^2}\left[-2y^2dx^+dx^- + \frac{1}{2}(y^4 - z^4)(dx^+)^2 + (dx^-)^2 + \frac{1}{2}z^2dz^2 - \frac{1}{2}y^2dy^2\right] \\ & + \mathcal{O}\left(1/\widehat{R}^4\right) . \end{aligned} \qquad (2.6)$$

When this metric is used to construct the worldsheet string action, the $O(1/\widehat{R}^0)$ terms lead to a quadratic (free) theory of worldsheet fields and the $O(1/\widehat{R}^2)$ terms lead to quartic interaction terms. We will give a perturbative treatment of the effect of these interactions on the energy spectrum of the leading free theory.

The Green-Schwarz action of type IIB superstring theory on $AdS_5 \times S^5$ can be expressed as a non-linear sigma model constructed from supersymmetric Cartan one-forms L_a^μ (the index a is a worldsheet index here) on the coset manifold $G/H = [SO(4,2) \times SO(6)]/[SO(4,1) \times SO(5)]$ [2]. The superconformal algebra of the coset manifold takes the generic form

$$[B_\mu, B_\nu] = f^\rho_{\mu\nu}B_\rho \qquad [F_\alpha, B_\nu] = f^\beta_{\alpha\nu}F_\beta \qquad \{F_\alpha, F_\beta\} = f^\mu_{\alpha\beta}B_\mu , \qquad (2.7)$$

where B_μ (F_α) represent bosonic (fermionic) generators. The Cartan forms L^μ and super-

connections L^α are determined by the structure constants $f^J_{\alpha\mu}$ and $f^\mu_{\alpha\beta}$ according to

$$L^\alpha_{at} = \left(\frac{\sinh t\mathcal{M}}{\mathcal{M}}\right)^\alpha_\beta (\mathcal{D}_a\theta)^\beta \tag{2.8}$$

$$L^\mu_{at} = e^\mu{}_\nu \partial_a x^\nu + 2\theta^\alpha f^\mu_{\alpha\beta}\left(\frac{\sinh^2(t\mathcal{M}/2)}{\mathcal{M}^2}\right)^\beta_\gamma (\mathcal{D}_a\theta)^\gamma \tag{2.9}$$

$$(\mathcal{M}^2)^\alpha_\beta = -\theta^\gamma f^\alpha_{\gamma\mu}\theta^\delta f^\mu_{\delta\beta}\ , \tag{2.10}$$

with $L^\mu_a = L^\mu_{at}|_{t=1}$. In $AdS_5\times S^5$, the Lagrangian takes the form

$$\mathcal{L}_{\rm Kin} = -\frac{1}{2}h^{ab}L^\mu_a L^\mu_b \tag{2.11}$$

$$\mathcal{L}_{\rm WZ} = -2i\epsilon^{ab}\int_0^1 dt\, L^\mu_{at} s^{IJ}\bar\theta^I\Gamma^\mu L^J_{bt}\ , \tag{2.12}$$

where Γ^μ are $SO(9,1)$ Dirac gamma matrices, $\eta_{\mu\nu}$ is the $SO(9,1)$ Minkowski metric and $s^{IJ}=\mathrm{diag}(1,-1)$. The worldsheet fermi fields θ^I $(I,J=1,2)$ are two $SO(9,1)$ Majorana–Weyl spinors of the same chirality $\Gamma_{11}\theta^I=\theta^I$. The κ-symmetry condition $\Gamma_0\Gamma_9\theta^I=\theta^I$ sets half the components of θ^I to zero. In constructing the Hamiltonian, it will be convenient to define $\psi=\sqrt{2}(\theta^1+i\theta^2)$: ψ_α is an eight-component complex spinor built out of the 16 components of θ^I that survive the gauge fixing. It will be useful to split components of ψ according to their transformation under the parity operator $\Pi\equiv\gamma^1\bar\gamma^2\gamma^3\bar\gamma^4$, where $\gamma^a,\bar\gamma^a$ are 8×8 $SO(8)$ gamma matrices.

Ultimately, we want to expand the GS worldsheet Hamiltonian in powers of $1/\widehat{R}$, along the lines of (2.6), in order to develop a perturbation theory of the string spectrum. To do this, we expand all the elements of the GS action (metric, spin connections, one-forms, etc.) in powers of $1/\widehat{R}$ and write the lightcone coordinates x^+ and x^- in terms of the physical transverse fields (by solving the x^- equations of motion and the covariant gauge constraints order-by-order). Explicit quantization rules are obtained by rewriting everything in terms of unconstrained canonical variables after removing the second-class constraints to which some of the fermionic canonical variables are subject. This awkward and rather complicated procedure is described in detail in [8, 9]. The final result for the Hamiltonian density, correct to $O(1/J)$, is

$$H = H_{\rm pp} + H_{\rm int}\ , \qquad H_{\rm int} = H_{\rm BB} + H_{\rm FF} + H_{\rm BF}\ , \tag{2.13}$$

where

$$H_{\rm pp} = \frac{1}{2}\left[(x^A)^2+(p_A)^2+(x'^A)^2\right] + \frac{i}{2}\left[\psi\psi' - \rho\rho' + 2\rho\Pi\psi\right]\ , \tag{2.14}$$

$$\begin{aligned} H_{\rm BB} = \frac{1}{\widehat{R}^2}\Bigg\{ & \frac{1}{4}\left[z^2\left(p_y^2+y'^2+2z'^2\right) - y^2\left(p_z^2+z'^2+2y'^2\right)\right] + \frac{1}{8}\left[(x^A)^2\right]^2 \\ & -\frac{1}{8}\left[\left[(p_A)^2\right]^2 + 2(p_A)^2(x'^A)^2 + \left[(x'^A)^2\right]^2\right] + \frac{1}{2}\left(x'^A p_A\right)^2\Bigg\}\ , \end{aligned} \tag{2.15}$$

$$
\begin{aligned}
H_{\rm FF} &= -\frac{1}{4\widehat{R}^2}\Big\{\left[(\psi'\psi)+(\rho\rho')\right](\rho\Pi\psi)-\frac{1}{2}(\psi'\psi)^2-\frac{1}{2}(\rho'\rho)^2+(\psi'\psi)(\rho'\rho)\\
&\quad+(\rho\psi')(\rho'\psi)-\frac{1}{2}\left[(\psi\rho')(\psi\rho')+(\psi'\rho)^2\right]+\Big[\frac{1}{12}(\psi\gamma^{jk}\rho)(\rho\gamma^{jk}\Pi\rho')\\
&\quad-\frac{1}{48}\left(\psi\gamma^{jk}\psi-\rho\gamma^{jk}\rho\right)\left(\rho'\gamma^{jk}\Pi\psi-\rho\gamma^{jk}\Pi\psi'\right)-(j,k\rightleftharpoons j',k')\Big]\Big\}\,, \qquad (2.16)
\end{aligned}
$$

$$
\begin{aligned}
H_{\rm BF} &= \frac{1}{\widehat{R}^2}\Big\{-\frac{i}{4}\left[(p_A)^2+(x'^A)^2+(y^2-z^2)\right](\psi\psi'-\rho\rho')\\
&\quad-\frac{1}{2}(p_Ax'^A)(\rho\psi'+\psi\rho')-\frac{i}{2}\left(p_k^2+y'^2-z^2\right)\rho\Pi\psi\\
&\quad+\frac{i}{4}(z'_jz_k)\left(\psi\gamma^{jk}\psi-\rho\gamma^{jk}\rho\right)-\frac{i}{4}(y'_{j'}y_{k'})\left(\psi\gamma^{j'k'}\psi-\rho\gamma^{j'k'}\rho\right)\\
&\quad-\frac{i}{8}(z'_ky_{k'}+z_ky'_{k'})\left(\psi\gamma^{kk'}\psi-\rho\gamma^{kk'}\rho\right)+\frac{1}{4}(p_ky_{k'}+z_kp_{k'})\psi\gamma^{kk'}\rho\\
&\quad+\frac{1}{4}(p_jz'_k)\left(\psi\gamma^{jk}\Pi\psi+\rho\gamma^{jk}\Pi\rho\right)-\frac{1}{4}(p_{j'}y'_{k'})\left(\psi\gamma^{j'k'}\Pi\psi+\rho\gamma^{j'k'}\Pi\rho\right)\\
&\quad-\frac{i}{2}(p_kp_{k'}-z'_ky'_{k'})\psi\gamma^{kk'}\Pi\rho\Big\}\,. \qquad (2.17)
\end{aligned}
$$

The canonical commutation relations of $p^A,\ x^A$ and $\psi,\ \rho$ mean that they can be expanded in bosonic creation operators $a_n^{A\dagger}$, fermionic creation operators $b_n^{\alpha\dagger}$ and their annihilation operator counterparts (where n is a mode index). Upper-case Latin letters $A,B,C,\cdots\in 1,\ldots,8$ indicate vectors of the transverse $SO(8)$: these are divided into the two $SO(4)$ subgroups which descend from the AdS_5 and S^5 subspaces. Lower-case letters $a,b,c,\cdots\in 1,\ldots,4$ label vectors in $SO(4)_{AdS}$, while $a',b',c',\cdots\in 5,\ldots,8$ indicate vectors in $SO(4)_{S^5}$. Spinors of $SO(9,1)$ are labeled by $\alpha,\beta,\gamma,\cdots\in 1,\ldots,8$. All told, there are 16 oscillators per mode n. The Fock space of physical string states is generated by acting on the ground state $|J\rangle$ (which carries the total angular momentum of the string state) with arbitrary combinations of the above creation operators. Matrix elements of the Hamiltonian are computed by expanding each of the fields and conjugate momenta of H in the mode creation and annihilation operators and acting on the Fock space in the obvious way.

The parameter equivalences between the string and gauge theories are determined by a modified AdS/CFT dictionary which emerges in the pp-wave/BMN limit. The R-charge is equated on the string side with $p_-\widehat{R}^2=J$, and the large R-charge limit corresponds to taking $J=p_-\widehat{R}^2\to\infty$ and $N_c\to\infty$, keeping the ratio N_c/J^2 fixed. The gauge theory coupling $\lambda=g_{YM}^2N_c$ is then replaced on the string side with the so-called modified 't Hooft coupling $\lambda'=g_{YM}^2N_c/J^2$.

At leading order in J^{-1}, H reduces to $H_{\rm pp}$ and the string theory reduces to a theory of eight free massive bosons and fermions. Altogether, the sixteen oscillators for each mode n contribute $\omega_n=\sqrt{1+n^2\lambda'}$ to the energy of the string and produce a highly degenerate

spectrum. For example, the 256 'two-impurity' states spanned by

$$\begin{array}{lll} a_n^{A\dagger} a_{-n}^{B\dagger} |J\rangle & b_n^{\alpha\dagger} b_{-n}^{\beta\dagger} |J\rangle & \text{(spacetime bosons)} \\ a_n^{A\dagger} b_{-n}^{\beta\dagger} |J\rangle & b_n^{\alpha\dagger} a_{-n}^{B\dagger} |J\rangle & \text{(spacetime fermions)} \end{array} \tag{2.18}$$

all have the same energy $E = 2\sqrt{1 + n^2\lambda'}$ in the large-J limit (note that the mode indices of the oscillators must sum to zero due to the level-matching constraint). This degeneracy is broken at the first non-leading order in J^{-1} by the action of the perturbing Hamiltonian $H_{\rm int} = H_{\rm BB} + H_{\rm FF} + H_{\rm BF}$.

To find the spectrum, one must diagonalize the 256×256 perturbation matrix schematically represented in table 1. The matrix of course block-diagonalizes on spacetime bosons

$(H)_{\rm int}$	$a_n^{A\dagger} a_{-n}^{B\dagger} \|J\rangle$	$b_n^{\alpha\dagger} b_{-n}^{\beta\dagger} \|J\rangle$	$a_n^{A\dagger} b_{-n}^{\alpha\dagger} \|J\rangle$	$a_{-n}^{A\dagger} b_n^{\alpha\dagger} \|J\rangle$
$\langle J\| a_n^A a_{-n}^B$	$H_{\rm BB}$	$H_{\rm BF}$	0	0
$\langle J\| b_n^\alpha b_{-n}^\beta$	$H_{\rm BF}$	$H_{\rm FF}$	0	0
$\langle J\| a_n^A b_{-n}^\alpha$	0	0	$H_{\rm BF}$	$H_{\rm BF}$
$\langle J\| a_{-n}^A b_n^\alpha$	0	0	$H_{\rm BF}$	$H_{\rm BF}$

Table 1: Complete Hamiltonian in the space of two-impurity string states

and spacetime fermions, so the problem is actually 128×128. The matrix elements of the non-vanishing sub-blocks are computed by expanding each of the fields (and conjugate momenta) in $H_{\rm BB}$, $H_{\rm FF}$ and $H_{\rm BF}$ in mode creation and annihilation operators and evaluating the indicated Fock space matrix elements. Finding the perturbed spectrum is then a matter of diagonalizing the explicit matrix constructed in this fashion. The calculation is algebraically tedious and requires the use of symbolic manipulation programs, but the end results are quite simple and verify, as previously described, a perfect match of the string spectrum to the gauge theory results out to two-loop order.

Our purpose in this paper is to work out the analogous results for higher-impurity states on both sides of the correspondence. In fact, we will limit our detailed considerations to the three-impurity problem, since it presents many interesting complications as compared to the two-impurity case, and it is not clear that any useful illumination will come from studying yet higher-impurity states.

3 Three-impurity string spectrum: leading order in λ'

The three-impurity Fock space block-diagonalizes into separate spacetime fermion and spacetime boson sectors. The bosonic sector contains states that are purely bosonic (composed of three bosonic string oscillators) and states with bi-fermionic components:

$$a_q^{A\dagger} a_r^{B\dagger} a_s^{C\dagger} |J\rangle \qquad a_q^{A\dagger} b_r^{\alpha\dagger} b_s^{\beta\dagger} |J\rangle \ . \tag{3.1}$$

Pure boson states are mixed by the bosonic sector of the Hamiltonian $H_{\rm BB}$, while states with bi-fermionic excitations are mixed both by the purely fermionic Hamiltonian $H_{\rm FF}$ and the bose-fermi sector $H_{\rm BF}$. The sector of spacetime fermion states is composed of purely fermionic excitations and mixed states containing two bosonic oscillators:

$$b_q^{\alpha\dagger} b_r^{\beta\dagger} b_s^{\gamma\dagger} \left|J\right\rangle \qquad a_q^{A\dagger} a_r^{B\dagger} b_s^{\alpha\dagger} \left|J\right\rangle \ . \tag{3.2}$$

Pure fermion states are acted on by $H_{\rm FF}$, and mixed states with bosonic excitations are acted on by $H_{\rm BB}$ and $H_{\rm BF}$. This block diagonalization of the perturbing Hamiltonian is displayed schematically in table 2.

$(H)_{int}$	$a^{A\dagger}a^{B\dagger}a^{C\dagger}\left\|J\right\rangle$	$a^{A\dagger}b^{\alpha\dagger}b^{\beta\dagger}\left\|J\right\rangle$	$b^{\alpha\dagger}b^{\beta\dagger}b^{\gamma\dagger}\left\|J\right\rangle$	$a^{A\dagger}a^{B\dagger}b^{\alpha\dagger}\left\|J\right\rangle$
$\left\langle J\right\| a^A a^B a^C$	H_{BB}	H_{BF}	0	0
$\left\langle J\right\| a^A b^\alpha b^\beta$	H_{BF}	$H_{FF}+H_{BF}$	0	0
$\left\langle J\right\| b^\alpha b^\beta b^\gamma$	0	0	H_{FF}	H_{BF}
$\left\langle J\right\| a^A a^B b^\alpha$	0	0	H_{BF}	$H_{BB}+H_{BF}$

Table 2: Three-impurity string states

The three-impurity string states are subject to the usual level-matching condition on the mode indices: $q+r+s=0$. There are two generically different solutions of this constraint: all mode indices different ($q\neq r\neq s$) and two indices equal (eg. $q=r=n,\ s=-2n$). In the inequivalent index case, there are $16^3=4,096$ degenerate states arising from different choices of spacetime labels on the mode creation operators. In the case of two equivalent indices, the dimension of the degenerate subspace is half as large (there are fewer permutations on mode indices that generate linearly independent states). The two types of basis break up into irreducible representations of $PSU(2,2|4)$ in different ways and must be studied separately.

As in the two-impurity case, the problem of diagonalizing the perturbation simplifies enormously when the matrix elements are expanded to leading order in λ'. We will take this approach here to obtain an overview of how degeneracies are lifted by the interaction. The generalization of the results to all loop orders in λ' (but still to first non-leading order in $1/J$) will be presented in the next section. We use the same methods, conventions and notations as in our recent detailed study of the two-impurity problem [9] (especially in Sec. 6 of that paper). It is once again the case that in the one-loop approximation, projection onto invariant subspaces under the manifest global $SO(4)\times SO(4)$ symmetry often diagonalizes the Hamiltonian directly (and at worst reduces it to a low-dimensional matrix). Symbolic manipulation programs were used to organize the complicated algebra and to perform explicit projections onto invariant subspaces.

3.1 Matrix evaluation: inequivalent mode indices ($q \neq r \neq s$)

In the sector of spacetime bosons, the subspace of purely bosonic states $a_q^{A\dagger} a_r^{B\dagger} a_s^{C\dagger} |J\rangle$ is 512-dimensional. When each of the three mode indices (q, r, s) are different, states with bi-fermionic excitations $a_q^{A\dagger} b_r^{\alpha\dagger} b_s^{\beta\dagger} |J\rangle$ are inequivalent under permutation of the mode indices, and form a 1,536-dimensional subsector. The entire bosonic sector of the three-impurity state space therefore contains 2,048 linearly independent states. The fermionic sector decomposes in a similar manner: the subsector of purely fermionic states $b_q^{\alpha\dagger} b_r^{\beta\dagger} b_s^{\gamma\dagger} |J\rangle$ is 512-dimensional; fermionic states containing two bosonic excitations $a_q^{A\dagger} a_r^{B\dagger} b_s^{\alpha\dagger} |J\rangle$ are inequivalent under permutation of the mode indices, and comprise an additional 1,536-dimensional subsector. Adding this 2,048-dimensional fermion sector brings the dimensionality of the entire state space to 4,096.

Our first task is to evaluate the interaction Hamiltonian matrix. The matrix elements needed to fill out the spacetime boson sector are listed in table 3. To evaluate the entries, we express the Hamiltonians (2.15-2.17) in terms of mode creation and annihilation operators, expand the result in powers of λ' and then compute the indicated matrix elements between three-impurity Fock space states. We collect below all the relevant results of this exercise.

$H_{\rm int}$	$a_s^{D\dagger} a_r^{E\dagger} a_q^{F\dagger} \|J\rangle$	$a_s^{D\dagger} b_r^{\gamma\dagger} b_q^{\delta\dagger} \|J\rangle$	$a_r^{D\dagger} b_q^{\gamma\dagger} b_s^{\delta\dagger} \|J\rangle$	$a_r^{D\dagger} b_s^{\gamma\dagger} b_q^{\delta\dagger} \|J\rangle$
$\langle J\| a_q^A a_r^B a_s^C$	$H_{\rm BB}$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm BF}$
$\langle J\| a_q^A b_r^\alpha b_s^\beta$	$H_{\rm BF}$	$H_{\rm FF} + H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm BF}$
$\langle J\| a_s^A b_q^\alpha b_r^\beta$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm FF} + H_{\rm BF}$	$H_{\rm BF}$
$\langle J\| a_r^A b_s^\alpha b_q^\beta$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm FF} + H_{\rm BF}$

Table 3: Interaction Hamiltonian on spacetime boson three-impurity string states ($q \neq r \neq s$)

We will use an obvious (m, n) matrix notation to distinguish the different entries in table 3. The purely bosonic, 512-dimensional $(1, 1)$ block has the explicit form

$$\begin{aligned}
\big\langle J\big| a_q^A a_r^B a_s^C (H_{\rm BB}) a_s^{D\dagger} a_r^{E\dagger} a_q^{F\dagger}\big| J\big\rangle \;&=\; \frac{\lambda'}{J}\delta^{AF}\delta^{BE}\delta^{CD}\left(rs + q(r+s) - q^2 - r^2 - s^2\right) \\
&+\frac{\lambda'}{2J}\Big\{\delta^{AF}\Big[(r^2+s^2)\big(\delta^{cd}\delta^{be} - \delta^{c'd'}\delta^{b'e'}\big) + (s^2-r^2)\big(\delta^{be}\delta^{c'd'} - \delta^{cd}\delta^{b'e'}\big) \\
&\qquad +2rs\big(\delta^{bd}\delta^{ce} - \delta^{bc}\delta^{de} - \delta^{b'd'}\delta^{c'e'} + \delta^{b'c'}\delta^{d'e'}\big)\Big] + \big(r \rightleftharpoons q,\ F \rightleftharpoons E,\ A \rightleftharpoons B\big) \\
&\qquad +\big(s \rightleftharpoons q,\ F \rightleftharpoons D,\ A \rightleftharpoons C\big)\Big\}\ . \qquad (3.3)
\end{aligned}$$

The off-diagonal entries that mix purely bosonic states $a_q^{A\dagger} a_r^{B\dagger} a_s^{C\dagger} |J\rangle$ with states containing bi-fermions $a_q^{A\dagger} b_r^{\alpha\dagger} b_s^{\beta\dagger} |J\rangle$ are given by a separate set of 512-dimensional matrices. The $(1, 2)$

block in table 3, for example, yields

$$\big\langle J\big|a_q^A a_r^B a_s^C (H_{\rm BF}) a_s^{D\dagger} b_r^{\alpha\dagger} b_q^{\beta\dagger}\big|J\big\rangle \;=\; \frac{\lambda'}{2J}\delta^{CD} qr\Big\{\big(\gamma^{ab'}\big)^{\alpha\beta} - \big(\gamma^{a'b}\big)^{\alpha\beta}\Big\}\,, \qquad (3.4)$$

where the index a (a') symbolizes the value of the vector index A, provided it is in the first (second) $SO(4)$. There are six blocks in this subsector, each given by a simple permutation of the mode indices (q,r,s) in eqn. (3.4). In table 3, these matrices occupy the $(1,2)$, $(1,3)$ and $(1,4)$ blocks, along with their transposes in the $(2,1)$, $(3,1)$ and $(4,1)$ entries.

The pure-fermion sector of the Hamiltonian, $H_{\rm FF}$, has non-vanishing matrix elements between states containing bi-fermionic excitations. The $H_{\rm FF}$ contribution to the $(2,2)$ block, for example, is given by

$$\begin{aligned}\big\langle J\big|b_q^\alpha b_r^\beta a_s^A (H_{\rm FF}) a_s^{B\dagger} b_r^{\gamma\dagger} b_q^{\delta\dagger}\big|J\big\rangle \;&=\; -\frac{\lambda'}{2J}(q-r)^2\delta^{AB}\delta^{\alpha\delta}\delta^{\gamma\beta} \\ &+\frac{\lambda'}{24J}\delta^{AB} qr\Big\{\big(\gamma^{ij}\big)^{\alpha\gamma}\,\big(\gamma^{ij}\big)^{\beta\delta} - \big(\gamma^{ij}\big)^{\alpha\beta}\big(\gamma^{ij}\big)^{\gamma\delta} - \big(\gamma^{ij}\big)^{\alpha\delta}\big(\gamma^{ij}\big)^{\beta\gamma} \\ &\quad -\big(\gamma^{i'j'}\big)^{\alpha\gamma}\big(\gamma^{i'j'}\big)^{\beta\delta} + \big(\gamma^{i'j'}\big)^{\alpha\beta}\big(\gamma^{i'j'}\big)^{\gamma\delta} + \big(\gamma^{i'j'}\big)^{\alpha\delta}\big(\gamma^{i'j'}\big)^{\beta\gamma}\Big\}\,. \qquad (3.5)\end{aligned}$$

A similar contribution, related to this one by simple permutations of the mode indices (q,r,s), appears in the diagonal blocks $(3,3)$ and $(4,4)$ as well.

The bose-fermi mixing Hamiltonian $H_{\rm BF}$ makes the following contribution to the lower diagonal blocks $(2,2)$, $(3,3)$ and $(4,4)$ in table 3:

$$\begin{aligned}\big\langle J\big|b_q^\alpha b_r^\beta a_s^A (H_{\rm BF}) a_s^{B\dagger} b_r^{\gamma\dagger} b_q^{\delta\dagger}\big|J\big\rangle \;&=\; \frac{\lambda'}{2J}\Big\{2s(q+r-s)\delta^{ab}\delta^{\alpha\delta}\delta^{\beta\gamma} - rs\Big[\big(\gamma^{ab}\big)^{\beta\gamma} - \big(\gamma^{a'b'}\big)^{\beta\gamma}\Big] \\ &\quad -sq\Big[\big(\gamma^{ab}\big)^{\alpha\delta} - \big(\,\gamma^{a'b'}\big)^{\alpha\delta}\Big] - 2\Big[q^2+r^2+s^2-s(q+r)\Big]\delta^{a'b'}\delta^{\alpha\delta}\delta^{\beta\gamma}\Big\}\,. \qquad (3.6)\end{aligned}$$

The $H_{\rm BF}$ sector also makes the following contribution to the off-diagonal $(2,3)$ block:

$$\big\langle J\big|b_q^\alpha b_r^\beta a_s^A (H_{\rm BF}) a_r^{B\dagger} b_q^{\gamma\dagger} b_s^{\delta\dagger}\big|J\big\rangle = -\frac{\lambda'}{2J}\delta^{\alpha\gamma} rs\Big\{\big(\delta^{ab}-\delta^{a'b'}\big)\delta^{\beta\delta} - \big(\gamma^{ab}\big)^{\beta\delta} + \big(\gamma^{a'b'}\big)^{\beta\delta}\Big\}\,. \quad (3.7)$$

The contributions of $H_{\rm BF}$ to the remaining off-diagonal blocks $(2,3)$, $(2,4)$, etc. are obtained by appropriate index permutations.

The sector of spacetime fermions decomposes in a similar fashion. The fermion analogue of table 3 for the bosonic sector appears in table 4. The $(1,1)$ fermion block is occupied by the pure-fermion sector of the Hamiltonian taken between the purely fermionic three-impurity

$H_{\rm int}$	$b_s^{\zeta\dagger}b_r^{\epsilon\dagger}b_q^{\delta\dagger}\left\vert J\right\rangle$	$a_s^{C\dagger}a_r^{D\dagger}b_q^{\delta\dagger}\left\vert J\right\rangle$	$a_r^{C\dagger}a_q^{D\dagger}b_s^{\delta\dagger}\left\vert J\right\rangle$	$a_r^{C\dagger}a_s^{D\dagger}b_q^{\delta\dagger}\left\vert J\right\rangle$
$\left\langle J\right\vert b_q^{\alpha}b_r^{\beta}b_s^{\gamma}$	$H_{\rm FF}$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm BF}$
$\left\langle J\right\vert b_q^{\alpha}a_r^{A}a_s^{B}$	$H_{\rm BF}$	$H_{\rm BB}+H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm BF}$
$\left\langle J\right\vert b_s^{\alpha}a_q^{A}a_r^{B}$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm BB}+H_{\rm BF}$	$H_{\rm BF}$
$\left\langle J\right\vert b_r^{\alpha}a_s^{A}a_q^{B}$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm BB}+H_{\rm BF}$

Table 4: Interaction Hamiltonian on spacetime fermion three-impurity states ($q\neq r\neq s$)

states $b_q^{\alpha\dagger}b_r^{\beta\dagger}b_s^{\gamma\dagger}\left\vert J\right\rangle$:

$$\begin{aligned}\left\langle J\right\vert b_q^{\alpha}b_r^{\beta}b_s^{\gamma}(H_{\rm FF})b_s^{\zeta\dagger}b_r^{\epsilon\dagger}b_q^{\delta\dagger}\left\vert J\right\rangle &= -\frac{\lambda'}{J}\left[q^2+r^2+s^2-rs-q(r+s)\right]\delta^{\alpha\delta}\delta^{\beta\epsilon}\delta^{\gamma\zeta}\\ &+\frac{\lambda'}{24J}\delta^{\alpha\delta}rs\Big\{\ \big(\gamma^{ij}\big)^{\beta\gamma}\big(\gamma^{ij}\big)^{\epsilon\zeta}-\big(\gamma^{ij}\big)^{\beta\epsilon}\big(\gamma^{ij}\big)^{\gamma\zeta}+\big(\gamma^{ij}\big)^{\beta\zeta}\big(\gamma^{ij}\big)^{\gamma\epsilon}\\ &-\big(\gamma^{i'j'}\big)^{\beta\gamma}\big(\gamma^{i'j'}\big)^{\epsilon\zeta}+\big(\gamma^{i'j'}\big)^{\beta\epsilon}\big(\gamma^{i'j'}\big)^{\gamma\zeta}-\big(\gamma^{i'j'}\big)^{\beta\zeta}\big(\gamma^{i'j'}\big)^{\gamma\epsilon}\\ &+\big(\ r\rightleftharpoons q,\ \alpha\rightleftharpoons\beta,\ \delta\rightleftharpoons\epsilon\big)+\big(s\rightleftharpoons q,\ \alpha\rightleftharpoons\gamma,\ \delta\rightleftharpoons\zeta\big)\Big\}\ .\end{aligned}\tag{3.8}$$

The off-diagonal $(1,2)$, $(1,3)$ and $(1,4)$ blocks (and their transposes) mix purely fermionic states with $a_s^{A\dagger}a_r^{B\dagger}b_q^{\alpha\dagger}\left\vert J\right\rangle$ states:

$$\left\langle J\right\vert b_q^{\alpha}b_r^{\beta}b_s^{\gamma}(H_{\rm BF})a_s^{A\dagger}a_r^{B\dagger}b_q^{\delta\dagger}\left\vert J\right\rangle = -\frac{\lambda'}{2J}\delta^{\alpha\delta}rs\Big\{\Big(\gamma^{ab'}\Big)^{\beta\gamma}-\Big(\gamma^{a'b}\Big)^{\beta\gamma}\Big\}\ .\tag{3.9}$$

The lower-diagonal $(2,2)$, $(3,3)$ and $(4,4)$ blocks receive contributions from the pure boson sector of the Hamiltonian:

$$\begin{aligned}\left\langle J\right\vert b_q^{\alpha}a_r^{A}a_s^{B}(H_{\rm BB})a_s^{C\dagger}a_r^{D\dagger}b_q^{\beta\dagger}\left\vert J\right\rangle &= -\frac{\lambda'}{2J}\delta^{\alpha\beta}\Big\{(r-s)^2\delta^{BC}\delta^{AD}-(r^2+s^2)\Big(\delta^{ad}\delta^{bc}-\delta^{a'd'}\delta^{b'c'}\Big)\\ &-2rs\Big(\delta^{ac}\delta^{bd}-\delta^{ab}\delta^{cd}\ -\delta^{a'c'}\delta^{b'd'}+\delta^{a'b'}\delta^{c'd'}\Big)+(r^2-s^2)\Big(\delta^{ad}\delta^{b'c'}-\delta^{a'd'}\delta^{bc}\Big)\Big\}\ .\end{aligned}\tag{3.10}$$

In the same diagonal blocks of table 4, the $H_{\rm BF}$ sector contributes

$$\begin{aligned}\left\langle J\right\vert b_q^{\alpha}a_r^{A}a_s^{B}(H_{\rm BF})a_s^{C\dagger}a_r^{D\dagger}b_q^{\beta\dagger}\left\vert J\right\rangle &= \frac{\lambda'}{8J}\Big\{\delta^{\alpha\beta}\Big[+\big(8q(r+s)-5(r^2+s^2)-6q^2\big)\delta^{AD}\delta^{BC}\\ &+(3q^2+s^2)\delta^{AD}\delta^{bc}\ +(3q^2+r^2)\delta^{BC}\delta^{ad}+(r^2-5q^2)\delta^{BC}\delta^{a'd'}+(s^2-5q^2)\delta^{AD}\delta^{b'c'}\Big]\\ &-4\delta^{BC}qr\Big[\ \big(\gamma^{ad}\big)^{\alpha\beta}-\big(\gamma^{a'd'}\big)^{\alpha\beta}\Big]-4\delta^{AD}qs\Big[\big(\gamma^{bc}\big)^{\alpha\beta}-\big(\gamma^{b'c'}\big)^{\alpha\beta}\Big]\Big\}\ .\end{aligned}\tag{3.11}$$

Finally, the off-diagonal blocks $(2,3)$, $(2,4)$ and $(3,4)$ (plus their transpose entries) are given by the $H_{\rm BF}$ matrix element

$$\begin{aligned}\left\langle J\middle|b_q^\alpha a_r^A a_s^B (H_{\rm BF}) a_r^{C\dagger} a_q^{D\dagger} b_s^{\beta\dagger}\middle|J\right\rangle &= -\frac{\lambda'}{32J}\delta^{AC}\Big\{\delta^{\alpha\beta}\Big[(q-s)^2\delta^{BD}-(q^2+14qs+s^2)\delta^{bd}\\ &\quad -(q^2-18qs+s^2)\delta^{b'd'}\Big]+16qs\Big[\left(\gamma^{bd}\right)^{\alpha\beta}-\left(\gamma^{b'd'}\right)^{\alpha\beta}\Big]\Big\}\ . \quad (3.12)\end{aligned}$$

A significant departure from the two-impurity case is that all these matrix elements have, along with their spacetime index structures, non-trivial dependence on the mode indices. The eigenvalues could potentially have very complicated mode-index dependence but, as we shall see, they do not. This amounts to a rigid consistency check on the whole procedure that was not present in the two-impurity case.

3.2 Matrix diagonalization: inequivalent mode indices ($q \neq r \neq s$)

We now turn to the task of diagonalizing the one-loop approximation to the perturbing Hamiltonian. To simplify the task, we exploit certain block diagonalizations that hold to leading order in λ' (but not to higher orders). While we eventually want to study the spectrum to all orders in λ', diagonalizing the Hamiltonian at one loop will reveal the underlying supermultiplet structure. As an example of the simplifications we have in mind, we infer from (3.4) that the matrix elements of $H_{\rm BF}$ between pure boson states $a_q^{A\dagger}a_r^{B\dagger}a_s^{C\dagger}\left|J\right\rangle$ and bifermionic spacetime bosons vanish to leading order in λ' if all three $SO(8)$ bosonic vector indices lie within the same $SO(4)$, descended either from AdS_5 or S^5. Restricting to such states brings the bosonic sector of the Hamiltonian into the block-diagonal form in table 5. This leaves two 64-dimensional subspaces of purely bosonic states on which the perturbation

$H_{\rm int}$	$a^{a\dagger}a^{b\dagger}a^{c\dagger}\left\|J\right\rangle + a^{a'\dagger}a^{b'\dagger}a^{c'\dagger}\left\|J\right\rangle$	$a^{A\dagger}b^{\alpha\dagger}b^{\beta\dagger}\left\|J\right\rangle$
$\left\langle J\right\| a^a a^b a^c + \left\langle J\right\| a^{a'} a^{b'} a^{c'}$	$H_{\rm BB}$	0
$\left\langle J\right\| a^A b^\alpha b^\beta$	0	$H_{\rm FF}+H_{\rm BF}$

Table 5: Block-diagonal $SO(4)$ projection on bosonic three-impurity string states

is block diagonal, as recorded in table 6.

$H_{\rm int}$	$a^{a\dagger}a^{b\dagger}a^{c\dagger}\left\|J\right\rangle$	$a^{a'\dagger}a^{b'\dagger}a^{c'\dagger}\left\|J\right\rangle$
$\left\langle J\right\| a^a a^b a^c$	$(H_{\rm BB})_{64\times 64}$	0
$\left\langle J\right\| a^{a'} a^{b'} a^{c'}$	0	$(H_{\rm BB})_{64\times 64}$

Table 6: SO(4) projection on purely bosonic states

Since the interaction Hamiltonian has manifest $SO(4) \times SO(4)$ symmetry, it is useful to project matrix elements onto irreps of that group before diagonalizing. In some cases the irrep is unique, and projection directly identifies the corresponding eigenvalue. In the cases where an irrep has multiple occurrences, there emerges an unavoidable matrix diagonalization that is typically of low dimension. In what follows, we will collect the results of carrying out this program on the one-loop interaction Hamiltonian. A very important feature of the results which appear is that all the eigenvalues turn out to have a common simple dependence on mode indices. More precisely, the expansion of the eigenvalues for inequivalent mode indices (q, r, s) out to first non-leading order in λ' and $1/J$ can be written as

$$E_J(q, r, s) = 3 + \frac{\lambda'(q^2 + r^2 + s^2)}{2} \left(1 + \frac{\Lambda}{J} + O(J^{-2})\right) , \tag{3.13}$$

where Λ is a pure number that characterizes the lifting of the degeneracy in the various sectors. The notation $\Lambda_{\rm BB}$, $\Lambda_{\rm BF}$ and $\Lambda_{\rm FF}$ will be used to denote energy corrections arising entirely from the indicated sectors of the perturbing Hamiltonian. This simple quadratic dependence of the eigenvalues on the mode indices does not automatically follow from the structure of the matrix elements themselves, but is important for the successful match to gauge theory eigenvalues. In what follows, we will catalog some of the different Λ values that occur, along with their $SO(4) \times SO(4)$ irreps (and multiplicities). When we have the complete list, we will discuss how they are organized into supermultiplets.

In the $SO(4)$ projection in table 6, we will find a set of 64 eigenvalues for both the $SO(4)_{AdS}$ and $SO(4)_{S^5}$ subsectors. We record this eigenvalue spectrum in table 7, using an $SU(2)^2 \times SU(2)^2$ notation. For comparison, it is displayed alongside the projection of the 2-impurity spectrum onto the same subspace (as found in [8, 9]). In the three-

$SO(4)_{AdS} \times SO(4)_{S^5}$	$\Lambda_{\rm BB}$
$(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$	-8
$[\mathbf{1},\mathbf{1};(\mathbf{2}+\mathbf{4}),\mathbf{2}] + [\mathbf{1},\mathbf{1};\mathbf{2},(\mathbf{2}+\mathbf{4})]$	-6
$[\mathbf{1},\mathbf{1};(\mathbf{2}+\mathbf{4}),(\mathbf{2}+\mathbf{4})]$	-4
$[(\mathbf{2}+\mathbf{4}),(\mathbf{2}+\mathbf{4});\mathbf{1},\mathbf{1}]$	-2
$[(\mathbf{2}+\mathbf{4}),\mathbf{2};\mathbf{1},\mathbf{1}] + [\mathbf{2},(\mathbf{2}+\mathbf{4});\mathbf{1},\mathbf{1}]$	0
$(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$	2

$SO(4)_{AdS} \times SO(4)_{S^5}$	$\Lambda_{\rm BB}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-6
$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	-4
$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3})$	-2
$(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$	-2
$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	0
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	2

Table 7: Three-impurity energy spectrum in the pure-boson $SO(4)$ projection (left panel) and two-impurity energy spectrum in the same projection (right panel)

impurity case, the $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$ level in the $SO(4)_{S^5}$ subsector clearly descends from the two-impurity singlet $(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$ in the same $SO(4)$ subgroup. In the same manner, the three-impurity $[\mathbf{1},\mathbf{1};(\mathbf{2}+\mathbf{4}),\mathbf{2}] + [\mathbf{1},\mathbf{1};\mathbf{2},(\mathbf{2}+\mathbf{4})]$ level descends from the $SO(4)_{S^5}$ antisymmetric two-impurity state $(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$, and the three-impurity $[\mathbf{1},\mathbf{1};(\mathbf{2}+\mathbf{4}),(\mathbf{2}+\mathbf{4})]$ level is tied to the two-impurity symmetric-traceless $(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3})$ irrep. In the $SO(4)_{S^5}$ sub-

sector, each of these levels receives a shift to the energy of -2. The total multiplicity of each of these levels is also increased by a factor of four when the additional $(\mathbf{2},\mathbf{2})$ is tensored into the two-impurity state space. The $SO(4)_{AdS}$ subsector follows a similar pattern: the $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$, $[(\mathbf{2}+\mathbf{4}),\mathbf{2};\mathbf{1},\mathbf{1}]+[\mathbf{2},(\mathbf{2}+\mathbf{4});\mathbf{1},\mathbf{1}]$ and $[(\mathbf{2}+\mathbf{4}),(\mathbf{2}+\mathbf{4});\mathbf{1},\mathbf{1}]$ levels appear as three-impurity descendants of the two-impurity irrep spectrum $(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})+(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})+(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$. In this subsector, however, the three-impurity energies are identical to those in the two-impurity theory.

The bosonic $SO(4)$ projection has a precise fermionic analogue. Similar to the bosons, the $SO(9,1)$ spinors $b_q^\dagger$ decompose as $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})+(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ under the action of Π parity:

$$\Pi\hat{b}_q^\dagger=\hat{b}_q^\dagger \qquad \Pi\tilde{b}_q^\dagger=-\tilde{b}_q^\dagger\ . \tag{3.14}$$

The notation $\hat{b}_q^\dagger$ labels $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ spinors with positive eigenvalue under Π, and $\tilde{b}_q^\dagger$ indicate $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ spinors which are negative under Π. Analogous to the $SO(4)$ projection on the $SO(8)$ bosonic operators $a_q^{A\dagger}\to a_q^{a\dagger}+a_q^{a'\dagger}$, projecting out the positive or negative eigenvalues of Π on the eight-component spinor $b_q^{\alpha\dagger}$ leaves a subspace of four-component spinors spanned by $\hat{b}_q^\dagger$ and $\tilde{b}_q^\dagger$.

We can perform a projection on the subsector in table 3 similar to that appearing in table 6. In this case, instead of three bosonic impurities mixing with a single bosonic (plus a bi-fermionic) excitation, we are now interested in projecting out particular interactions between a purely fermionic state and a state with one fermionic and two bosonic excitations. Using $\pm$ to denote the particular representation of the fermionic excitations, the off-diagonal elements given by (3.9) vanish for $+++\to\pm$ and $---\to\pm$ interactions. In other words, the pure fermion states in the $(1,1)$ block of table 4 will not mix with states containing two bosonic excitations if all three fermionic oscillators lie in the same Π projection. This projection appears schematically in table 8.

$H_{\rm int}$	$\hat{b}^{\alpha\dagger}\hat{b}^{\beta\dagger}\hat{b}^{\gamma\dagger}\left\vert J\right\rangle+\tilde{b}^{\alpha\dagger}\tilde{b}^{\beta\dagger}\tilde{b}^{\gamma\dagger}\left\vert J\right\rangle$	$a^{A\dagger}a^{B\dagger}b^{\alpha\dagger}\left\vert J\right\rangle$
$\left\langle J\right\vert\hat{b}^{\alpha}\hat{b}^{\beta}\hat{b}^{\gamma}+\left\langle J\right\vert\tilde{b}^{\alpha}\tilde{b}^{\beta}\tilde{b}^{\gamma}$	$H_{\rm FF}$	0
$\left\langle J\right\vert a^A a^B b^\alpha$	0	$H_{\rm BB}+H_{\rm BF}$

Table 8: Block-diagonal projection on fermionic three-impurity string states

The $(1,1)$ pure fermion block in table 8 breaks into two 64-dimensional subsectors under this projection. By tensoring an additional $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ or $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ impurity into the two-impurity state space, we expect to see a multiplicity structure in this projection given by

$$\begin{aligned}
(\mathbf{1},\mathbf{2})\times(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2}) &= (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})+[\mathbf{1},\mathbf{2};\mathbf{1},(\mathbf{2}+\mathbf{4})]\\
&\quad+[\mathbf{1},(\mathbf{2}+\mathbf{4});\mathbf{1},\mathbf{2}]+[\mathbf{1},(\mathbf{2}+\mathbf{4});\mathbf{1},(\mathbf{2}+\mathbf{4})]\ ,\\
(\mathbf{2},\mathbf{1})\times(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1}) &= (\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})+[\mathbf{2},\mathbf{1};(\mathbf{2}+\mathbf{4}),\mathbf{1}]\\
&\quad+[(\mathbf{2}+\mathbf{4}),\mathbf{1};\mathbf{2},\mathbf{1}]+[(\mathbf{2}+\mathbf{4}),\mathbf{1};(\mathbf{2}+\mathbf{4}),\mathbf{1}]\ ,
\end{aligned} \tag{3.15}$$

for a total of 128 states. The projections onto the two 64-dimensional Π_+ and Π_- subspaces yield identical eigenvalues and multiplicities. The results for both subspaces are presented in table 9: The two-impurity bi-fermion states in table 9 are spacetime bosons while the

$SO(4)_{AdS} \times SO(4)_{S^5}$	$\Lambda_{\rm FF}$
$(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$	-3
$[\mathbf{2},\mathbf{1};(\mathbf{2}+\mathbf{4}),\mathbf{1}] + [\mathbf{1},\mathbf{2};\mathbf{1},(\mathbf{2}+\mathbf{4})]$	-1
$[(\mathbf{2}+\mathbf{4}),\mathbf{1};\mathbf{2},\mathbf{1}] + [\mathbf{1},(\mathbf{2}+\mathbf{4});\mathbf{1},\mathbf{2}]$	-5
$[(\mathbf{2}+\mathbf{4}),\mathbf{1};(\mathbf{2}+\mathbf{4}),\mathbf{1}] + [\mathbf{1},(\mathbf{2}+\mathbf{4});\mathbf{1},(\mathbf{2}+\mathbf{4})]$	-3

$SO(4)_{AdS} \times SO(4)_{S^5}$	$\Lambda_{\rm FF}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	0
$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	-4
$(\mathbf{3},\mathbf{1};\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{3})$	-2

Table 9: Spectrum of three-impurity states (left panel) and two-impurity states (right panel) created by $\Pi_\pm$-projected fermionic creation operators

tri-fermion states are spacetime fermions. For comparison purposes, we have displayed both spectra. Note that the $O(1/J)$ energy corrections of the two types of state are simply displaced by -1 relative to each other.

This exhausts the subspaces that can be diagonalized by simple irrep projections. The remaining eigenvalues must be obtained by explicit diagonalization of finite dimensional submatrices obtained by projection onto representations with multiple occurrence. The upshot of these more complicated eigenvalue calculations is that the first-order λ' eigenvalues take on all integer values from $\Lambda = -8$ to $\Lambda = +2$, alternating between spacetime bosons and fermions as Λ is successively incremented by one unit.

3.3 Assembling eigenvalues into supermultiplets

Finally, we need to understand how the perturbed three-impurity spectrum breaks up into extended supersymmetry multiplets. This is relatively easy to infer from the multiplicities of the perturbed eigenvalues (and the multiplicities are a side result of the calculation of the eigenvalues themselves). In the last subsection, we described a procedure for diagonalizing the one-loop perturbing Hamiltonian on the $4,096$-dimensional space of three-impurity string states with mode indices $p \neq q \neq r$. The complete results for the eigenvalues Λ and their multiplicities are stated in table 10 (we use the notation of (3.13), while the B and F subscripts are used to indicate bosonic and fermionic levels in the supermultiplet).

Λ	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
Multiplicity	4_B	40_F	180_B	480_F	840_B	1008_F	840_B	480_F	180_B	40_F	4_B

Table 10: Complete three-impurity energy spectrum (with multiplicities)

The eigenvalues Λ must meet certain conditions if the requirements of $PSU(2,2|4)$ symmetry are to be met. The eigenvalues in question are lightcone energies and thus dual to the gauge theory quantity $\Delta = D - J$, the difference between scaling dimension and R-charge. Since conformal invariance is part of the full symmetry group, states are organized into conformal multiplets built on conformal primaries. A supermultiplet will contain several conformal primaries having the same value of Δ and transforming into each other under the supercharges. All 16 supercharges increment the dimension of an operator by 1/2, but only 8 of them (call them $\mathcal{Q}_\alpha$) also increment the R-charge by 1/2, so as to leave Δ unchanged. These 8 supercharges act as 'raising operators' on the conformal primaries of a supermultiplet: starting from a super-primary of lowest R-charge, the other conformal primaries are created by acting on it in all possible ways with the eight $\mathcal{Q}_\alpha$. Primaries obtained by acting with L factors of $\mathcal{Q}_\alpha$ on the super-primary are said to be at level L in the supermultiplet (since the $\mathcal{Q}_\alpha$ anticommute, the range is $L = 0$ to $L = 8$). The multiplicities of states at the various levels are also determined: for each $L = 0$ primary operator, there will be C_L^8 such operators at level L (where C_m^n is the binomial coefficient). If the $L = 0$ primary has multiplicity s, summing over all L gives $2^8 s = 256s$ conformal primaries in all.

These facts severely restrict the quantity Λ in the general expression (3.13) above. Although the states in the degenerate multiplet all have the same J, they actually belong to different levels L in more than one supermultiplet. A state of given L is a member of a supermultiplet built on a 'highest-weight' or super-primary state with $R = J - L/2$. Since all the primaries in a supermultiplet have the same Δ, the joint dependence of eigenvalues on λ, J, L must be of the form $\Delta(\lambda, J - L/2)$. The only way the expansion of (3.13) can be consistent with this is if $\Lambda = L + c$, where c is a pure numerical constant (recall that $\lambda' = \lambda/J^2$). Successive members of a supermultiplet must therefore have eigenvalues separated by exactly 1 and the difference between 'top' ($L = 8$) and 'bottom' ($L = 0$) eigenvalues for Λ must be exactly 8.

The Λ eigenvalues in table 10 are integer-spaced, which is consistent with supersymmetry requirements. However, because the range between top and bottom eigenvalues is 10, rather than 8, the 4,096-dimensional space must be built on more than one type of extended supermultiplet, with more than one choice of c in the general formula $\Lambda = L + c$. This is to be contrasted with the two-impurity case, where the degenerate space was exactly 256-dimensional and was spanned by a single superconformal primary whose lowest member was a singlet under both Lorentz transformations and the residual $SO(4)$ R-symmetry. We can readily infer what superconformal primaries are needed to span the degenerate three-impurity state space by applying a little numerology to table 10. The lowest eigenvalue is $\Lambda = -8$: it has multiplicity 4 and, according to table 7, its $SO(4) \times SO(4)$ decomposition is $(\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{2})$ (spacetime scalar, R-charge $SO(4)$ four-vector). According to the general arguments about how the full extended supermultiplet is built by acting on a 'bottom' state with the eight raising operators, it is the base of a supermultiplet of 4×256 states extending up to $\Lambda = 0$. By the same token, there is a highest eigenvalue $\Lambda = +2$: it has multiplicity 4 and, according to table 7, its $SO(4) \times SO(4)$ decomposition is $(\mathbf{2}, \mathbf{2}; \mathbf{1}, \mathbf{1})$ (spacetime vector, R-charge singlet). Using lowering operators instead of raising operators, we see that one derives from it a

supermultiplet of 4×256 operators with eigenvalues extending from $\Lambda = -6$ to $\Lambda = +2$. The multiplicities of the Λ eigenvalues occurring in these two supermultiplets are of course given by binomial coefficients, as described above. By comparing with the *total* multiplicities of each allowed Λ (as listed in table 10) we readily see that what remains are 8×256 states with eigenvalues running from $\Lambda = -7$ to $\Lambda = +1$ with the correct binomial coefficient pattern of multiplicities. The top and bottom states here are spacetime fermions and must lie in a spinor representation of the Lorentz group. It is not hard to see that they lie in the eight-dimensional $SO(4) \times SO(4)$ irrep $(\mathbf{2}, \mathbf{1}; \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}; \mathbf{2}, \mathbf{1})$. This exhausts all the states and we conclude that the three-impurity state space is spanned by three distinct extended superconformal multiplets. The detailed spectrum is given in table 11 (where the last line records the total multiplicity at each level as given in table 10 and the first line records the two-impurity spectrum for reference). Note the peculiar feature that certain energies are shared by all three multiplets: this is an accidental degeneracy that does not survive at higher loop order.

Λ	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	
$\Delta_0 = 2$			1_B	8_F	28_B	56_F	70_B	56_F	28_B	8_F	1_B	scalar
$\Delta_0 = 3$	4	32	112	224	280	224	112	32	4			$SO(4)_{S^5}$ vector
			4	32	112	224	280	224	112	32	4	$SO(4)_{AdS_5}$ vector
		8	64	224	448	560	448	224	64	8		spinor
Total	4_B	40_F	180_B	480_F	840_B	1008_F	840_B	480_F	180_B	40_F	4_B	4,096

Table 11: Submultiplet breakup of the three-impurity spectrum

A complete analysis of the agreement with gauge theory anomalous dimensions will have to be deferred until a later section: the dimensions of three-impurity gauge theory operators are much harder to calculate than those of the two-impurity operators and there are few results in the literature, even at one loop. However, it is worth making a few preliminary remarks at this point. Since there are three superconformal multiplets, we have only three independent anomalous dimensions to compute. Minahan and Zarembo [4] found that the problem simplifies dramatically if we study the one-loop anomalous dimension of the special subset of single-trace operators of the form $\mathrm{tr}\,(\phi^I Z^J)$ (and all possible permutations of the fields inside the trace), where the R-charge is carried by an $SO(4) \times SO(4)$ singlet scalar field Z and the impurities are insertions of a scalar field ϕ lying in the $(\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{2})$ (vector) irrep of the residual $SO(4) \times SO(4)$ symmetry. More formally, these operators are in the $SO(4) \times SO(4)$ irrep obtained by completely symmetrizing I vectors in the $(\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{2})$ irrep. The crucial point is that such operators form a 'closed sector', mixing *only* among themselves under the anomalous dimension operator. More importantly, the action of the one-loop anomalous dimension operator on this closed sector can be recast as the action of an integrable spin chain Hamiltonian of a type solvable by Bethe ansatz techniques. Although

the Bethe ansatz is generally not analytically soluble, Minahan and Zarembo used it to obtain a virial expansion for the anomalous dimension in which the number I of impurities is held fixed, while the R-charge J is taken to be large (see eqn. (5.29) in [4]). In terms of the number of spin chain lattice sites $\mathfrak{L}$, their result appears as

$$\gamma_{\mathfrak{so}(6)} = \frac{\lambda}{2\mathfrak{L}^3} \sum_n M_n k_n^2 \left(\mathfrak{L} + M_n + 1\right) + O(\mathfrak{L}^{-4}) \ . \tag{3.16}$$

The integer k_n represents pseudoparticle momenta on the spin chain, and is dual to the string theory worldsheet mode indices; the quantity M_n labels the number of trace impurities with identical k_n. With I impurities, the spin chain length is given in terms of the R-charge by $\mathfrak{L} = J + I$, which leads to

$$\gamma_{\mathfrak{so}(6)} = \frac{\lambda}{2J^3} \sum_n M_n k_n^2 \left(J - 2I + M_n + 1\right) + O(J^{-4}) \ . \tag{3.17}$$

This virial expansion is similar in character to (3.13) and, for $I = 3$ (the three-impurity case), it matches that equation precisely with $\Lambda = -4$.

On the string theory side, three completely symmetrized $(\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{2})$ vectors form a tensor in the $(\mathbf{1}, \mathbf{1}; \mathbf{4}, \mathbf{4})$ irrep; such an irrep can be constructed from three $SO(4)_{S^5}$ vector (bosonic) creation operators. Table 7 shows that the corresponding string perturbation theory eigenvalue is (at one-loop order) $\Lambda = -4$ as well. We infer from table 11 that this eigenvalue lies at level $L = 4$ of the $SO(4)_{S^5}$ vector superconformal multiplet (and this argument takes care of the gauge theory/string theory comparison for all other operators in that multiplet).

The sector described above is often called an $\mathfrak{so}(6)^2$ sector on the gauge theory side, with reference to the subalgebra of the full superconformal algebra under which it is invariant. In an $\mathfrak{su}(2)$ subspace of the $\mathfrak{so}(6)$, this sector becomes closed to all loop order. For future reference, we note that Beisert [12] has identified two other 'closed sectors' of operators in the gauge theory. In addition to the bosonic $\mathfrak{su}(2)$ sector, a bosonic $\mathfrak{sl}(2)$ sector and an $\mathfrak{su}(2|3)$ sector (of which the closed $\mathfrak{su}(2)$ sector is a subgroup) are also exactly closed. It should be noted that integrable $\mathfrak{sl}(2)$ spin chains were discovered some time ago in phenomenologically-motivated studies of the scaling behavior of high-energy scattering amplitudes in physical, non-supersymmetric QCD [13] (see also [14, 15, 16, 17]). The $\mathfrak{su}(2|3)$ spin chain was studied more recently in [6]: this closed sector breaks into the $\mathfrak{su}(2)$ bosonic sector and a fermionic subsector which, to avoid confusion, we simply denote as a subgroup of $\mathfrak{su}(2|3)$.

In the string theory, the subsectors analogous to the gauge theory $\mathfrak{sl}(2)$ and $\mathfrak{su}(2|3)$ are constructed out of completely symmetrized $SO(4)_{AdS}$ bosons and completely symmetrized fermions of the same Π eigenvalue, respectively [9]. They correspond to the central $L = 4$ levels of the remaining two supermultiplets in table 11, and a calculation of their eigenvalues would complete the analysis of the match between three-impurity operators and string states

[2]This notation is used to distinguish the protected gauge theory symmetry groups from those in the string theory.

at one-loop order. Unfortunately, explicit general results for three-impurity operator dimensions, analogous to those obtained by Minahan and Zarembo for the $\mathfrak{so}(6)$ sector [4], have not been obtained for the other closed sectors. The Bethe ansatz for the general one-loop integrable spin chain presented in [18] could easily be exploited for this purpose.[3] However, since we eventually want to go beyond one-loop, where Bethe ansatz technology is less well-developed, we have found it more useful to develop numerical methods for evaluating spin chain eigenvalues (we refer the reader to [11] for a check of our results against Bethe-ansatz techniques, including the higher-loop corrections of [19]). This subject will be developed in a later section.

3.4 Two equivalent mode indices ($q = r = n,\ s = -2n$)

When two mode indices are allowed to be equal, the analysis becomes slightly more complicated. Since we are diagonalizing a Hamiltonian that is quartic in oscillators in a basis of three-impurity string states, one oscillator in the "in" state must always be directly contracted with one oscillator in the "out" state and, with two equal mode indices, there are many more nonvanishing contributions to each matrix element. While the matrix elements are more complicated, the state space is only half as large when two mode indices are allowed to be equal (only half as many mode-index permutations on the basis states generate linearly independent states). As a result, the fermionic and bosonic sectors of the Hamiltonian are are each 1,024-dimensional. By the same token, the multiplet structure of the energy eigenstates will be significantly different from the unequal mode index case studied in the previous subsection.

To study this case, we make the mode index choice

$$q = r = n \qquad s = -2n\ . \tag{3.18}$$

The structure of matrix elements of the string Hamiltonian between spacetime bosons is given in table 12. This table seems to describe a 3×3 block matrix with 512×512 blocks

$H_{\rm int}$	$a_{-2n}^{D\dagger} a_n^{E\dagger} a_n^{F\dagger} \lvert J\rangle$	$a_{-2n}^{D\dagger} b_n^{\gamma\dagger} b_n^{\delta\dagger} \lvert J\rangle$	$a_n^{D\dagger} b_n^{\gamma\dagger} b_{-2n}^{\delta\dagger} \lvert J\rangle$
$\langle J\rvert\, a_n^A a_n^B a_{-2n}^C$	$H_{\rm BB}$	$H_{\rm BF}$	$H_{\rm BF}$
$\langle J\rvert\, a_n^A b_n^\alpha b_{-2n}^\beta$	$H_{\rm BF}$	$H_{\rm FF} + H_{\rm BF}$	$H_{\rm BF}$
$\langle J\rvert\, a_{-2n}^A b_n^\alpha b_n^\beta$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm FF} + H_{\rm BF}$

Table 12: Bosonic three-impurity string perturbation matrix with ($q = r = n,\ s = -2n$)

in each subsector, giving a 1,536-dimensional state space. However, the vector and spinor indices are required to run over values that generate linearly independent basis states. This

[3] We are indebted to N. Beisert for this observation.

eliminates one third of the possible index assignments, implying that the matrix is in fact $1,024 \times 1,024$.

To evaluate the entries in table 12, we express the Hamiltonians (2.15-2.17) in terms of mode creation and annihilation operators, expand the result in powers of λ' and compute the indicated matrix elements between three-impurity Fock space states. We collect below all the relevant results of this exercise for this equal-mode-index case.

The purely bosonic subsector in the $(1,1)$ block is given by

$$\begin{aligned}
\langle J|a_n^A a_n^B a_{-2n}^C (H_{\rm BB}) a_{-2n}^{D\dagger} a_n^{E\dagger} a_n^{F\dagger}\,|J\rangle = \frac{n^2\,\lambda}{2J}\Big\{ & 5\,\delta^{BF}\,\delta^{cd}\delta^{ae} + 5\,\delta^{AF}\,\delta^{cd}\delta^{be} - 4\,\delta^{BF}\,\delta^{ad}\delta^{ce} \\
& +4\,\delta^{BF}\,\delta^{ac}\delta^{de} + 4\,\delta^{AF}\,\delta^{bc}\delta^{de} + 5\,\delta^{BE}\,\delta^{cd}\delta^{af} - 4\,\delta^{BE}\,\delta^{ad}\delta^{cf} + 4\,\delta^{BE}\,\delta^{ac}\delta^{df} + 4\,\delta^{AE}\,\delta^{bc}\delta^{df} \\
& -4\,\delta^{bd}\Big(\delta^{AF}\,\delta^{ce} + \delta^{AE}\,\delta^{cf}\Big) + 3\,\delta^{BF}\,\delta^{ae}\,\delta^{c'd'} + 3\,\delta^{AF}\,\delta^{be}\,\delta^{c'd'} + 3\,\delta^{BE}\,\delta^{af}\,\delta^{c'd'} \\
& -3\,\delta^{BF}\,\delta^{cd}\,\delta^{a'e'} - 3\,\delta^{AF}\,\delta^{cd}\,\delta^{b'e'} - 5\,\delta^{BF}\,\delta^{c'd'}\delta^{a'e'} - 5\,\delta^{AF}\,\delta^{c'd'}\delta^{b'e'} + 4\,\delta^{BF}\,\delta^{a'd'}\delta^{c'e'} \\
& +4\,\delta^{AF}\,\delta^{b'd'}\delta^{c'e'} - 4\,\delta^{BF}\,\delta^{a'c'}\delta^{d'e'} - 4\,\delta^{AF}\,\delta^{b'c'}\delta^{d'e'} - 3\,\delta^{BE}\,\delta^{cd}\,\delta^{a'f'} - 3\,\delta^{AE}\,\delta^{cd}\,\delta^{b'f'} \\
& -5\,\delta^{BE}\,\delta^{c'd'}\delta^{a'f'} - 5\,\delta^{AE}\,\delta^{c'd'}\delta^{b'f'} + 4\,\delta^{BE}\,\delta^{a'd'}\delta^{c'f'} + 4\,\delta^{AE}\,\delta^{b'd'}\delta^{c'f'} - 4\,\delta^{BE}\,\delta^{a'c'}\delta^{d'f'} \\
& -4\,\delta^{AE}\,\delta^{b'c'}\delta^{d'f'} + \delta^{AE}\,\delta^{bf}\Big(5\,\delta^{cd} + 3\,\delta^{c'd'}\Big) - 2\,\delta^{CD}\Big[9\Big(\delta^{BE}\delta^{AF} + \delta^{AE}\delta^{BF}\Big) - \delta^{be}\delta^{af} \\
& -\delta^{ae}\delta^{bf} + \delta^{ab}\delta^{ef} + \delta^{b'e'}\delta^{a'f'} + \delta^{a'e'}\delta^{b'f'} - \delta^{a'b'}\delta^{e'f'}\Big]\Big\}\,. \qquad (3.19)
\end{aligned}$$

This matrix element exhibits the same antisymmetry between the $SO(4)_{AdS}$ and $SO(4)_{S^5}$ indices that is exhibited in eqn. (3.3). The off-diagonal $H_{\rm BF}$ mixing sector is essentially equivalent to its counterpart in eqn. (3.4):

$$\Big\langle J\Big|a_n^A a_n^B a_{-2n}^C (H_{\rm BF}) a_{-2n}^{D\dagger} b_n^{\alpha\dagger} b_n^{\beta\dagger}\Big|J\Big\rangle \;=\; \frac{n^2\lambda'}{2J}\delta^{CD}\Big\{\Big(\gamma^{ab'}\Big)^{\alpha\beta} - \Big(\gamma^{a'b}\Big)^{\alpha\beta}\Big\}\,. \qquad (3.20)$$

The diagonal contributions from the pure fermion sector $H_{\rm FF}$ in the $(2,2)$ and $(3,3)$ blocks of table 12 appear as

$$\begin{aligned}
\Big\langle J\Big|b_n^\alpha b_n^\beta a_{-2n}^A (H_{\rm FF}) a_{-2n}^{B\dagger} b_n^{\gamma\dagger} b_n^{\delta\dagger}\Big|J\Big\rangle \;=\; & \frac{n^2\lambda'}{24J}\delta^{AB}\Big\{\big(\gamma^{ij}\big)^{\alpha\gamma}\big(\gamma^{ij}\big)^{\beta\delta} - \big(\gamma^{ij}\big)^{\alpha\beta}\big(\gamma^{ij}\big)^{\gamma\delta} \\
& -\big(\gamma^{ij}\big)^{\alpha\delta}\big(\gamma^{ij}\big)^{\beta\gamma} - \big(\gamma^{i'j'}\big)^{\alpha\gamma}\big(\gamma^{i'j'}\big)^{\beta\delta} + \big(\gamma^{i'j'}\big)^{\alpha\beta}\big(\gamma^{i'j'}\big)^{\gamma\delta} + \big(\gamma^{i'j'}\big)^{\alpha\delta}\big(\gamma^{i'j'}\big)^{\beta\gamma}\Big\}\,. \qquad (3.21)
\end{aligned}$$

The $H_{\rm BF}$ sector exhibits the following contribution to the lower diagonal blocks $(2,2)$ and $(3,3)$:

$$\begin{aligned}
\Big\langle J\Big|b_n^\alpha b_n^\beta a_{-2n}^A (H_{\rm BF}) a_{-2n}^{B\dagger} b_n^{\gamma\dagger} b_n^{\delta\dagger}\Big|J\Big\rangle \;=\; & \frac{n^2\lambda'}{J}\Big\{-10\,\delta^{a'b'}\left(\delta^{\alpha\delta}\delta^{\beta\gamma} - \delta^{\alpha\gamma}\delta^{\beta\delta}\right) \\
& -8\,\delta^{ab}\left(\delta^{\alpha\delta}\delta^{\beta\gamma} - \delta^{\alpha\gamma}\delta^{\beta\delta}\right) - \delta^{\alpha\gamma}\Big[(\gamma^{ab})^{\beta\delta} - (\gamma^{a'b'})^{\beta\delta}\Big] + \delta^{\alpha\delta}\Big[(\gamma^{ab})^{\beta\gamma} - (\gamma^{a'b'})^{\beta\gamma}\Big] \\
& +\delta^{\beta\gamma}\Big[\,(\gamma^{ab})^{\alpha\delta} - (\gamma^{a'b'})^{\alpha\delta}\Big] - \delta^{\beta\delta}\Big[(\gamma^{ab})^{\alpha\gamma} - (\gamma^{a'b'})^{\alpha\gamma}\Big]\Big\}\,. \qquad (3.22)
\end{aligned}$$

Finally, the off-diagonal version of (3.22) appears in the $(2,3)$ block (along with its transpose in the $(3,2)$ block):

$$\begin{aligned}\left\langle J\middle|b_n^\alpha b_n^\beta a_{-2n}^A(H_{\rm BF})a_n^{B\dagger}b_n^{\gamma\dagger}b_{-2n}^{\delta\dagger}\middle|J\right\rangle &= -\frac{n^2\lambda'}{J}\Big\{\delta^{a'b'}\left(\delta^{\alpha\delta}\delta^{\beta\gamma}-\delta^{\alpha\gamma}\delta^{\beta\delta}\right)-\delta^{ab}\left(\delta^{\alpha\delta}\delta^{\beta\gamma}-\delta^{\alpha\gamma}\delta^{\beta\delta}\right)\\ &\quad+\delta^{\alpha\gamma}\Big[\ (\gamma^{ab})^{\beta\delta}-(\gamma^{a'b'})^{\beta\delta}\Big]-\delta^{\beta\gamma}\Big[(\gamma^{ab})^{\alpha\delta}-(\gamma^{a'b'})^{\alpha\delta}\Big]\Big\}\ .\end{aligned}\tag{3.23}$$

The fermionic sector perturbation matrix is displayed schematically in table 13. Like table 12, it is $1,024\times 1,024$ once redundant index assignments are eliminated.

$H_{\rm int}$	$b_{-2n}^{\zeta\dagger}b_n^{\epsilon\dagger}b_n^{\delta\dagger}\,\vert J\rangle$	$a_{-2n}^{C\dagger}a_n^{D\dagger}b_n^{\delta\dagger}\,\vert J\rangle$	$a_n^{C\dagger}a_n^{D\dagger}b_{-2n}^{\delta\dagger}\,\vert J\rangle$
$\langle J\vert\, b_n^\alpha b_n^\beta b_{-2n}^\gamma$	$H_{\rm FF}$	$H_{\rm BF}$	$H_{\rm BF}$
$\langle J\vert\, b_n^\alpha a_n^A a_{-2n}^B$	$H_{\rm BF}$	$H_{\rm BB}+H_{\rm BF}$	$H_{\rm BF}$
$\langle J\vert\, b_{-2n}^\alpha a_n^A a_n^B$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm BB}+H_{\rm BF}$

Table 13: Fermionic three-impurity string perturbation matrix ($q=r=n,\ s=-2n$)

The purely fermionic subsector in the $(1,1)$ block of table 13 takes the form

$$\begin{aligned}\left\langle J\middle|b_n^\alpha b_n^\beta b_{-2n}^\gamma(H_{\rm FF})b_{-2n}^{\zeta\dagger}b_n^{\epsilon\dagger}b_n^{\delta\dagger}\middle|J\right\rangle &= \frac{9n^2\lambda'}{J}\delta^{\gamma\zeta}\left(\delta^{\alpha\epsilon}\delta^{\beta\delta}-\delta^{\alpha\delta}\delta^{\beta\epsilon}\right)\\ &\quad+\frac{n^2\lambda'}{24J}\Big\{\delta^{\gamma\zeta}\Big[(\gamma^{ij})^{\alpha\beta}(\gamma^{ij})^{\delta\epsilon}\ -(\gamma^{ij})^{\alpha\delta}(\gamma^{ij})^{\beta\epsilon}+(\gamma^{ij})^{\alpha\epsilon}(\gamma^{ij})^{\beta\delta}-(\gamma^{i'j'})^{\alpha\beta}(\gamma^{i'j'})^{\delta\epsilon}\\ &\qquad+(\gamma^{i'j'})^{\alpha\delta}(\gamma^{i'j'})^{\beta\epsilon}\ -(\gamma^{i'j'})^{\alpha\epsilon}(\gamma^{i'j'})^{\beta\delta}\Big]-2\delta^{\alpha\delta}\Big[(\gamma^{ij})^{\beta\gamma}(\gamma^{ij})^{\epsilon\zeta}-(\gamma^{ij})^{\beta\epsilon}(\gamma^{ij})^{\gamma\zeta}\\ &\qquad+(\gamma^{ij})^{\beta\zeta}(\gamma^{ij})^{\gamma\epsilon}\ -(\gamma^{i'j'})^{\beta\gamma}(\gamma^{i'j'})^{\epsilon\zeta}+(\gamma^{i'j'})^{\beta\epsilon}(\gamma^{i'j'})^{\gamma\zeta}-(\gamma^{i'j'})^{\beta\zeta}(\gamma^{i'j'})^{\gamma\epsilon}\Big]\\ &\quad+2\delta^{\alpha\epsilon}\Big[(\gamma^{ij})^{\beta\gamma}(\gamma^{ij})^{\delta\zeta}\ -(\gamma^{ij})^{\beta\delta}(\gamma^{ij})^{\gamma\zeta}+(\gamma^{ij})^{\beta\zeta}(\gamma^{ij})^{\gamma\delta}-(\gamma^{i'j'})^{\beta\gamma}(\gamma^{i'j'})^{\delta\zeta}\\ &\qquad+(\gamma^{i'j'})^{\beta\delta}(\gamma^{i'j'})^{\gamma\zeta}\ -(\gamma^{i'j'})^{\beta\zeta}(\gamma^{i'j'})^{\gamma\delta}\Big]+2\delta^{\beta\delta}\Big[(\gamma^{ij})^{\alpha\gamma}(\gamma^{ij})^{\epsilon\zeta}-(\gamma^{ij})^{\alpha\epsilon}(\gamma^{ij})^{\gamma\zeta}\\ &\qquad+(\gamma^{ij})^{\alpha\zeta}(\gamma^{ij})^{\gamma\epsilon}\ -(\gamma^{i'j'})^{\alpha\gamma}(\gamma^{i'j'})^{\epsilon\zeta}+(\gamma^{i'j'})^{\alpha\epsilon}(\gamma^{i'j'})^{\gamma\zeta}-(\gamma^{i'j'})^{\alpha\zeta}(\gamma^{i'j'})^{\gamma\epsilon}\Big]\\ &\quad-2\delta^{\beta\epsilon}\Big[(\gamma^{ij})^{\alpha\gamma}(\gamma^{ij})^{\delta\zeta}\ -(\gamma^{ij})^{\alpha\delta}(\gamma^{ij})^{\gamma\zeta}+(\gamma^{ij})^{\alpha\zeta}(\gamma^{ij})^{\gamma\delta}-(\gamma^{i'j'})^{\alpha\gamma}(\gamma^{i'j'})^{\delta\zeta}\\ &\qquad+(\gamma^{i'j'})^{\alpha\delta}(\gamma^{i'j'})^{\gamma\zeta}\ -(\gamma^{i'j'})^{\alpha\zeta}(\gamma^{i'j'})^{\gamma\delta}\Big]\Big\}\ .\end{aligned}\tag{3.24}$$

The off-diagonal blocks $(1,2)$ and $(1,3)$ receive contributions from the $H_{\rm BF}$ sector:

$$\begin{aligned}\left\langle J\middle|b_n^\alpha b_n^\beta b_{-2n}^\gamma(H_{\rm BF})a_{-2n}^{A\dagger}a_n^{B\dagger}b_n^{\delta\dagger}\middle|J\right\rangle &= \frac{n^2\lambda'}{J}\Big\{\delta^{\alpha\delta}\Big[\left(\gamma^{ab'}\right)^{\beta\gamma}-\left(\gamma^{a'b}\right)^{\beta\gamma}\Big]\\ &\quad-\delta^{\delta\beta}\Big[\left(\gamma^{ab'}\right)^{\alpha\gamma}-\left(\gamma^{a'b}\right)^{\alpha\gamma}\Big]\Big\}\ .\end{aligned}\tag{3.25}$$

The bosonic sector $H_{\rm BB}$ contributes to the $(2,2)$ and $(3,3)$ blocks:

$$\begin{aligned}\left\langle J\left|b_q^\alpha a_r^A a_s^B (H_{\rm BB}) a_s^{C\dagger} a_r^{D\dagger} b_q^{\beta\dagger}\right|J\right\rangle &= -\frac{n^2\lambda'}{2J}\delta^{\alpha\beta}\Big\{9\,\delta^{AD}\delta^{BC}+4\,\delta^{ac}\delta^{bd}-4\,\delta^{ab}\delta^{cd}\\ &-\delta^{ad}\left(5\,\delta^{bc}+3\,\delta^{b'c'}\right)-4\,\delta^{a'c'}\delta^{b'd'}+4\,\delta^{a'b'}\delta^{c'd'}+\delta^{a'd'}\left(5\,\delta^{b'c'}+3\,\delta^{bc}\right)\Big\}\,. \end{aligned} \tag{3.26}$$

In the same lower-diagonal blocks, $H_{\rm BF}$ exhibits the contribution

$$\begin{aligned}\left\langle J\left|b_n^\alpha a_n^A a_{-2n}^B (H_{\rm BF}) a_{-2n}^{C\dagger} a_n^{D\dagger} b_n^{\beta\dagger}\right|J\right\rangle &= -\frac{n^2\lambda'}{8J}\Big\{39\,\delta^{\alpha\beta}\delta^{AD}\delta^{BC}+\delta^{\alpha\beta}\delta^{AD}\left(\delta^{b'c'}-7\,\delta^{bc}\right)\\ &-4\,\delta^{\alpha\beta}\delta^{BC}\left(\delta^{ad}-\delta^{a'd'}\right)+4\,\delta^{BC}\Big[(\gamma^{ad})^{\alpha\beta}-(\gamma^{a'd'})^{\alpha\beta}\Big]-8\,\delta^{AD}\Big[(\gamma^{bc})^{\alpha\beta}-(\gamma^{b'c'})^{\alpha\beta}\Big]\Big\}\,. \end{aligned} \tag{3.27}$$

Finally, $H_{\rm BF}$ yields matrix elements in the off-diagonal block $(2,3)$:

$$\begin{aligned}\left\langle J\left|b_n^\alpha a_n^A a_{-2n}^B (H_{\rm BF}) a_n^{C\dagger} a_n^{D\dagger} b_{-2n}^{\beta\dagger}\right|J\right\rangle &= -\frac{n^2\lambda'}{32J}\Big\{9\,\delta^{\alpha\beta}\delta^{AC}\delta^{BD}+9\,\delta^{\alpha\beta}\delta^{AD}\delta^{BC}\\ &+\delta^{\alpha\beta}\delta^{AC}\left(23\delta^{bd}-41\delta^{b'd'}\right)+\delta^{\alpha\beta}\delta^{AD}\left(23\delta^{bc}-41\delta^{b'c'}\right)\\ &-32\delta^{AD}\Big[(\gamma^{bc})^{\alpha\beta}-(\gamma^{b'c'})^{\alpha\beta}\Big]-32\delta^{AC}\Big[(\gamma^{bd})^{\alpha\beta}-(\gamma^{b'd'})^{\alpha\beta}\Big]\Big\}\,. \end{aligned} \tag{3.28}$$

We can perform a full symbolic diagonalization of the $1,024\times 1,024$ bosonic and fermionic perturbation matrices to obtain the one-loop in λ', $O(1/J)$ energy corrections. They can all be expressed in terms of dimensionless eigenvalues Λ according to the standard formula (3.13) modified by setting $q=r=n,\ s=-2n$:

$$E_J(n) = 3 + 3n^2\lambda'\left(1+\frac{\Lambda}{J}+O(J^{-2})\right)\,. \tag{3.29}$$

The resulting spectrum is displayed in table 14. The levels clearly organize themselves into

Λ_1 (S^5 vector)	$-23/3$	$-20/3$	$-17/3$	$-14/3$	$-11/3$	$-8/3$	$-5/3$	$-2/3$	$1/3$
Multiplicity	4_B	32_F	112_B	224_F	280_B	224_F	112_B	32_F	4_B
Λ_2 (AdS_5 vector)	$-19/3$	$-16/3$	$-13/3$	$-10/3$	$-7/3$	$-4/3$	$-1/3$	$2/3$	$5/3$
Multiplicity	4_B	32_F	112_B	224_F	280_B	224_F	112_B	32_F	4_B

Table 14: Spectrum of three-impurity string Hamiltonian with $(q=r=n,\ s=-2n)$

two superconformal multiplets built on vector primary states. Note that the spinor multiplet

is absent and that the degeneracy between multiplets that was seen in the inequivalent mode index case has been lifted. The spinor multiplet is absent for the following reason: it contains a representation at level $L = 4$ arising from fermion creation operators completely symmetrized on $SO(4) \times SO(4)$ spinor indices; such a construct must vanish unless all the creation operator mode indices are different.

If we keep track of the $SO(4) \times SO(4)$ irrep structure, we find that the symmetric-traceless bosonic $SO(4)_{S^5}$ states arising from the closed $\mathfrak{su}(2)$ subsector fall into the $-11/3$ $[280_B]$ level. This is the counterpart of the -4 $[280_B]$ level in table 11. To compare with Minahan and Zarembo's Bethe ansatz calculation of the corresponding gauge theory operator dimension, we must evaluate eqn. (3.17) with the appropriate choice of parameters. In particular, $M_n = 2$ when two mode indices are allowed to coincide and, comparing with eqn. (3.29), we find perfect agreement with the string theory prediction $\Lambda = -11/3$. States at level $L = 4$ in the second multiplet in table 14 correspond to operators in the $\mathfrak{sl}(2)$ closed sector of the gauge theory and the eigenvalue $\Lambda = -7/3$ $[280_B]$ amounts to a prediction for the one-loop anomalous dimension of that class of gauge theory operators. As mentioned at the end of the previous subsection, we will need to develop a numerical treatment of the $\mathfrak{sl}(2)$ spin chain Hamiltonian in order to assess the agreement between string theory and gauge theory in this sector.

4 Three impurity string spectrum: all orders in λ'

In the previous section, we have studied the eigenvalue spectrum of the string theory perturbation Hamiltonian expanded to leading order in $1/J$ and to one-loop order in λ'. The expansion in λ' was for convenience only since our expressions for matrix elements are exact in this parameter. We should, in principle, be able to obtain results that are exact in λ' (but still of leading order in $1/J$). This is a worthwhile enterprise since recent progress on the gauge theory side has made it possible to evaluate selected operator anomalous dimensions to two- and three-loop order. The simple one-loop calculations of the previous sections have given us an overview of how the perturbed string theory eigenvalues are organized into superconformal multiplets. This provides a very useful orientation for the more complex all-orders calculation, to which we now turn.

4.1 Inequivalent mode indices: $(q \neq r \neq s)$

Our first step is to collect the exact matrix elements of the perturbing Hamiltonian between three-impurity states of unequal mode indices. The block structure of the perturbation matrix in the spacetime boson sector is given in table 3 and the exact form of the $(1, 1)$

block is

$$\langle J|a_q^A a_r^B a_s^C (H_{\rm BB}) a_s^{D\dagger} a_r^{E\dagger} a_q^{F\dagger}\,|J\rangle = -\frac{1}{2\omega_q\omega_r\omega_s}\Bigg\{\delta^{BE}\omega_r\Big[\delta^{CD}\delta^{AF}(s^2+q^2(1+2s^2\lambda'))$$
$$-(q^2+s^2)\delta^{cd}\delta^{af} - 2qs(\delta^{ad}\delta^{cf}-\delta^{ac}\delta^{df}) + (q^2-s^2)\delta^{af}\delta^{c'd'} - (q^2-s^2)\delta^{a'f'}\delta^{cd}$$
$$+(q^2+s^2)\delta^{c'd'}\delta^{a'f'} + 2qs(\delta^{a'd'}\delta^{c'f'}-\delta^{a'c'}\delta^{d'f'})\Big] + \Big(C\rightleftharpoons B,\ D\rightleftharpoons E,\ s\rightleftharpoons r\Big)$$
$$+\Big(A\rightleftharpoons B,\ F\rightleftharpoons E,\ q\rightleftharpoons r\Big)\Bigg\}\,, \tag{4.1}$$

where we define $\omega_q \equiv \sqrt{q^2+1/\lambda'}$ to simplify this and other similar expressions.

The off-diagonal $H_{\rm BF}$ contributions to the $(1,2)$, $(1,3)$ and $(1,4)$ blocks are yet more complicated. To simplify the expressions, we define

$$F_1 \equiv \sqrt{(\omega_q+q)(\omega_r-r)} \qquad F_2 \equiv \sqrt{(\omega_q-q)(\omega_r+r)}$$
$$F_3 \equiv \sqrt{(\omega_q-q)(\omega_r-r)} \qquad F_4 \equiv \sqrt{(\omega_q+q)(\omega_r+r)}\,. \tag{4.2}$$

Using these functions, the matrix elements in these off-diagonal subsectors are given by:

$$\langle J|a_q^A a_r^B a_s^C (H_{\rm BF}) a_s^{D\dagger} b_r^{\alpha\dagger} b_q^{\beta\dagger}\,|J\rangle = \frac{\delta^{CD}}{32\omega_q\omega_r J}\Bigg\{\frac{8}{\sqrt{\lambda'}}(F_1-F_2)\delta^{AB}\delta^{\gamma\delta} - 2(q-r)(F_3+F_4)\delta^{AB}\delta^{\gamma\delta}$$
$$+4(q-r)(F_3+F_4)(\gamma^{ab})^{\gamma\delta} - 2(q+r)(F_3-F_4)(\gamma^{ab'})^{\gamma\delta}$$
$$+(2qF_3-2qF_4+2rF_3-2rF_4)(\gamma^{a'b})^{\gamma\delta} - (4qF_3+4qF_4-4rF_3-4rF_4)(\gamma^{a'b'})^{\gamma\delta}$$
$$+\frac{8}{\sqrt{\lambda'}}(F_2-F_1)\delta^{a'b'}\delta^{\gamma\delta} + 4(q-r)(F_3+F_4)\delta^{\gamma\delta}\delta^{a'b'} - 2(q-r)(F_3+F_4)\delta^{\gamma\delta}(\delta^{ab}-\delta^{a'b'})$$
$$-4\lambda'\omega_q\omega_r(q-r)(F_3+F_4)\delta^{AB}\delta^{\gamma\delta} + 4\sqrt{\lambda'}(qr-\omega_q\omega_r)\Big[(F_1+F_2)\Big((\gamma^{ab'})^{\gamma\delta}-(\gamma^{a'b})^{\gamma\delta}\Big)$$
$$-(F_1-F_2)\delta^{\gamma\delta}(\delta^{ab}-\delta^{a'b'})\Big] + 2(\omega_q+\omega_r)(F_3+F_4)\Big[(\gamma^{ab'})^{\gamma\delta}-(\gamma^{a'b})^{\gamma\delta}\Big]$$
$$+4\sqrt{\lambda'}(r\omega_q-q\omega_r)(F_1+F_2)\Big[(\gamma^{ab})^{\gamma\delta}-(\gamma^{a'b'})^{\gamma\delta}\Big]$$
$$-4\lambda'(q-r)(F_3-F_4)(r\omega_q+q\omega_r)\delta^{AB}\delta^{\gamma\delta} - \lambda'\delta^{AB}$$
$$+2\lambda'(\omega_q\omega_r-qr)(q-r)(F_3+F_4)\delta^{AB}\delta^{\gamma\delta} + 4\sqrt{\lambda'}(\omega_q\omega_r+qr)(F_1-F_2)\delta^{AB}\delta^{\gamma\delta}$$
$$-2\lambda'(q-r)(\omega_q\omega_r+qr)(F_3+F_4)\delta^{AB}\delta^{\gamma\delta}\Bigg\}\,. \tag{4.3}$$

The $H_{\rm FF}$ contribution to the lower-diagonal blocks $(2,2)$, $(3,3)$ and $(4,4)$ is

$$\begin{aligned}
&\langle J|b_q^\alpha b_r^\beta a_s^A(\mathrm{H_{FF}})\mathrm{a_s^{B\dagger}b_r^{\gamma\dagger}b_q^{\delta\dagger}}\,|\mathrm{J}\rangle = \\
&\quad\frac{\delta^{AB}}{48\omega_r\omega_s J}\sqrt{\lambda'}\Big\{2rs\sqrt{1/\lambda'}\Big[\Big((\gamma^{ij})^{\alpha\gamma}(\gamma^{ij})^{\beta\delta}-(\gamma^{i'j'})^{\alpha\gamma}(\gamma^{i'j'})^{\beta\delta}\Big)\\
&\quad-\Big((\gamma^{ij})^{\alpha\delta}(\gamma^{ij})^{\beta\gamma}-(\gamma^{i'j'})^{\alpha\delta}(\gamma^{i'j'})^{\beta\gamma}\Big)-\Big((\gamma^{ij})^{\alpha\beta}(\gamma^{ij})^{\gamma\delta}-(\gamma^{i'j'})^{\alpha\beta}(\gamma^{i'j'})^{\gamma\delta}\Big)\Big]\\
&\quad-12\Big[2\delta^{\alpha\delta}\delta^{\beta\gamma}\Big(s^2\sqrt{1/\lambda'}-2rs\sqrt{\lambda'}\omega_r\omega_s+r^2(2s^2\sqrt{\lambda'}+\sqrt{1/\lambda'})\Big)\Big]\Big\}\,.
\end{aligned}\tag{4.4}$$

The bose-fermi Hamiltonian $H_{\rm BF}$ contributes the following matrix elements to the same lower-diagonal blocks:

$$\begin{aligned}
\langle J|b_q^\alpha b_r^\beta a_s^A(H_{\rm BF})a_s^{B\dagger}b_r^{\gamma\dagger}b_q^{\delta\dagger}\,|J\rangle &= -\frac{1}{2\omega_q\omega_r\omega_s}\Big\{s\sqrt{\lambda'}\delta^{ab}\delta^{\alpha\delta}\delta^{\beta\gamma}\Big[s\omega_r(2q^2\sqrt{\lambda'}+\sqrt{1/\lambda'})\\
&+s\omega_q(2r^2\sqrt{\lambda'}+\sqrt{1/\lambda'})-2\omega_q\omega_r\omega_s(q+r)\sqrt{\lambda'}\Big]+\delta^{a'b'}\delta^{\alpha\delta}\delta^{\beta\gamma}\Big[2\omega_r q^2(1+s^2\lambda')\\
&+s^2\omega_r+2\omega_q r^2(1+s^2\lambda')+s^2\omega_q-2s(q+r)\lambda'\omega_q\omega_r\omega_s\Big]\\
&+sr\omega_q\delta^{\alpha\delta}\Big[(\gamma^{ab})^{\beta\gamma}-(\gamma^{a'b'})^{\beta\gamma}\Big]+sq\omega_r\delta^{\beta\gamma}\Big[(\gamma^{ab})^{\alpha\delta}-(\gamma^{a'b'})^{\alpha\delta}\Big]\Big\}\,.
\end{aligned}\tag{4.5}$$

To simplify off-diagonal elements in the $(2,3)$, $(2,4)$ and $(3,4)$ blocks, we define

$$\begin{aligned}
G_1 &\equiv \sqrt{(\omega_r+r)(\omega_s-s)} \qquad & G_2 &\equiv \sqrt{(\omega_r-r)(\omega_s+s)}\\
G_3 &\equiv \sqrt{(\omega_r-r)(\omega_s-s)} \qquad & G_4 &\equiv \sqrt{(\omega_r+r)(\omega_s+s)}\,.
\end{aligned}\tag{4.6}$$

The matrix elements in these subsectors are then given by

$$\begin{aligned}
&\langle J|b_q^\alpha b_r^\beta a_s^A(H_{\rm BF})a_r^{B\dagger}b_q^{\gamma\dagger}b_s^{\delta\dagger}\,|J\rangle = \\
&\quad-\frac{1}{16(\lambda'\omega_r\omega_s)^{3/2}}\Big\{\sqrt{\omega_r\omega_s}\lambda'\delta^{\alpha\gamma}\Big[2\delta^{ab}\delta^{\beta\delta}\Big[(G_1+G_2)(2-2\lambda'\omega_r\omega_s)\\
&\quad+(r+s)\sqrt{\lambda'}\Big(G_4-\lambda' G_4(r-\omega_r)(s-\omega_s)+G_3(-1+rs\lambda'+r\omega_s\lambda'+\omega_r(s+\omega_s)\lambda')\Big)\Big]\\
&\quad+2\sqrt{\lambda'}\Big[(r+s)(G_3-G_4)+\sqrt{\lambda'}(G_1-G_2)(r\omega_s-s\omega_r)\Big]\Big[(\gamma^{ab})^{\beta\delta}-(\gamma^{a'b'})^{\beta\delta}\Big]\\
&\quad+\sqrt{\lambda'}\Big[2rs\sqrt{\lambda'}G_1-2rs\sqrt{\lambda'}G_2+(r-s)(G_3+G_4)+(\omega_s-\omega_r)(G_3-G_4)\\
&\quad+2\omega_r\omega_s\sqrt{\lambda'}(G_2-G_1)\Big]\Big[(\gamma^{ab'})^{\beta\delta}-(\gamma^{a'b})^{\beta\delta}\Big]+2\delta^{a'b'}\delta^{\beta\delta}\Big[-2rs\sqrt{\lambda'}(G_1-G_2)\\
&\quad+(r+s)\sqrt{\lambda'}\Big(-G_4-\lambda' G_4(r-\omega_r)(s-\omega_s)\\
&\quad+G_3(1+rs\lambda'+r\omega_s\lambda'+\omega_r(s+\omega_s)\lambda')\Big)\Big]\Big]\Big\}\,.
\end{aligned}\tag{4.7}$$

The entries in the spacetime fermion block matrix of table 8 are far too complicated to write out explicitly: they are best generated, viewed and manipulated with computer algebra techniques. The explicit formulas, along with a collection of the Mathematica programs written to generate and work with them, are available on the web [20].

We were not able to symbolically diagonalize the complete perturbation matrix built from the exact (in λ') matrix elements listed above: with the computing resources available to us, the routines for diagonalizing the full 2,048-dimensional matrices would not terminate in any reasonable time. As noted in the previous section, however, gauge theory arguments suggest that there are three protected $SO(4) \times SO(4)$ irreps that do not mix with any other irreps. It is a straightforward matter to project the perturbation matrix onto these unique protected irreps to obtain analytic expressions for the corresponding exact eigenvalues. In fact, the superconformal multiplet structure of the three-impurity problem is such that the energies/dimensions of all other irreps can be inferred from those of the three protected irreps. Hence, this method will give us exact expressions for all the energy levels of the three-impurity problem.

Consider first the $\mathfrak{sl}(2)$ closed sector. The dual sector is generated on the string theory side by bosonic creation operators completely symmetrized (and traceless) on $SO(4)_{AdS}$ vector indices. The simplest way to make this projection on eqn. (4.1) is to compute diagonal elements between the symmetrized states

$$a_q^{(a\dagger} a_r^{b\dagger} a_s^{c\dagger)} \, |J\rangle \ , \tag{4.8}$$

with $a \neq b \neq c$ (and, of course, $a, b, c \in 1, \ldots, 4$). The charges of the fermionic oscillators under this subgroup are $\pm 1/2$, so the three-boson state of this type cannot mix with one boson and two fermions (or any other state). Hence, the above projection of eqn. (4.1) yields the closed sector eigenvalue correction

$$\begin{aligned} \delta E_{AdS}(q,r,s,J) &= \frac{1}{J\omega_q\omega_r\omega_s}\Big\{ qs(1-qs\lambda')\omega_r + qr(1-qr\lambda')\omega_s + rs(1-rs\lambda')\omega_q \\ &\qquad + [qr + s(q+r)]\,\lambda'\omega_q\omega_r\omega_s \Big\} \\ &\approx \frac{1}{J}\Big\{ -2(q^2+qr+r^2)\lambda' - \frac{15}{8}\big(q^2r^2(q+r)^2\big)\lambda'^3 + \ldots \Big\} \ . \end{aligned} \tag{4.9}$$

To facilitate eventual comparison with gauge theory results, we have performed a small-λ' expansion in the final line with the substitution $s \to -(q+r)$ (since the mode indices satisfy the constraint $s + q + r = 0$). The leading correction $-2(q^2 + qr + r^2)\lambda'$ reproduces the one-loop eigenvalue $\Lambda_{\rm BB} = -2$ $[280_B]$ located at level $L = 4$ in the $SO(4)_{AdS}$ multiplet in table 11.

The closed $\mathfrak{su}(2)$ sector is generated by bosonic creation operators completely symmetrized on traceless $SO(4)_{S^5}$ indices. Projection onto this irrep is most simply achieved by choosing all mode operators in eqn. (4.1) to carry symmetrized, traceless $SO(4)_{S^5}$ labels

(they can also be thought of as carrying charge +1 under some $SO(2)$ subgroup of $SO(4)_{S^5}$). Direct projection yields the $SO(4)_{S^5}$ eigenvalue

$$\begin{aligned}\delta E_{S^5}(q,r,s,J) &= -\frac{1}{J\omega_q\omega_r\omega_s}\Big\{\big[qr+r^2+q^2(1+r^2\lambda')\big]\,\omega_s+\big[qs+s^2+q^2(1+s^2\lambda')\big]\,\omega_r \\ &\qquad +\big[rs+s^2+r^2(1+s^2\lambda')\big]\,\omega_q-\big[rs+q(r+s)\big]\,\lambda'\omega_q\omega_r\omega_s\Big\} \\ &\approx \frac{1}{J}\Big\{-4(q^2+qr+r^2)\lambda'+(q^2+qr+r^2)^2\lambda'^2 \\ &\qquad -\frac{3}{4}\big(q^6+3q^5r+8q^4r^2+11q^3r^3+8q^2r^4+3qr^5+r^6\big)\lambda'^3+\dots\Big\}\,. \qquad (4.10)\end{aligned}$$

This is the all-loop formula corresponding to gauge theory operator dimensions in the closed $\mathfrak{su}(2)$ subsector; the leading-order term $-4(q^2+qr+r^2)\lambda'$ reproduces the one-loop eigenvalue $\Lambda_{\rm BB}=-4$ $[280_B]$ at level $L=4$ in the $SO(4)_{S^5}$ vector multiplet in table 11.

The eigenvalue of the symmetrized pure-fermion irrep can be obtained by evaluating the exact matrix element $H_{\rm FF}$ acting on three symmetrized fermionic creation operators with $SO(4)\times SO(4)$ indices chosen to lie in the same Π projection (with inequivalent mode indices). The exact energy shift for this irrep turns out to be

$$\begin{aligned}\delta E_{\rm Fermi}(q,r,s,J) &= -\frac{1}{4\,J\omega_q\omega_r\omega_s}\Big\{-4\big(rs+q(r+s)\big)\lambda'\omega_q\omega_r\omega_s \\ &\qquad +\Big[\omega_q\left(2s^2+4r^2s^2\lambda'+2r^2\right)+\big(s\to r,\ r\to q,\ q\to s\big)+\big(q\rightleftharpoons r\big)\Big]\Big\} \\ &\approx \frac{1}{J}\Big\{-3(q^2+qr+r^2)\lambda'+\frac{1}{2}(q^2+qr+r^2)^2\lambda'^2 \\ &\qquad -\frac{3}{16}\big(2q^6+6q^5r+21q^4r^2+32q^3r^3+21q^2r^4+6qr^5+2r^6\big)\lambda'^3+\dots\Big\}\,. \qquad (4.11)\end{aligned}$$

The leading-order λ' correction $-3(q^2+qr+r^2)\lambda'$ reproduces the $\Lambda_{\rm FF}=-3$ $[580_F]$ eigenvalue at the $L=4$ level in the spinor multiplet in table 11. This and the higher order terms in the eigenvalue will eventually be compared with the dimensions of operators in the closed, fermionic $\mathfrak{su}(2|3)$ sector in the gauge theory.

The argument we are making relies heavily on the claim that the perturbation matrix is block diagonal on the closed subsectors described above: we have evaluated the exact energy shift on these subsectors by simply taking the diagonal matrix element of the perturbing Hamiltonian in a particular state in each sector. We will now carry out a simple numerical test of the claimed block diagonalization of the full perturbing Hamiltonian. The basic idea is that, while it is impractical to algebraically diagonalize the full $2,048\times 2,048$ perturbation matrices, it is quite easy to do a numerical diagonalization for a specific choice of λ' and mode indices q,r,s. One can then check that the numerical eigenvalues match the analytic predictions evaluated at the chosen coupling and mode indices. For definiteness, we choose

$$q=1 \qquad r=2 \qquad s=-3 \qquad \lambda'=1\,. \qquad (4.12)$$

The predicted eigenvalue shifts of the three protected states, evaluated at the parameter choices of (4.12) are given in table 15. These values come directly from eqns. (4.9,4.10,4.11) above (with J set to unity, for convenience). Since we want to compare these energies

$\delta E:\ \lambda'=1$		$q=1,\ r=2,\ s=-3$
$\delta E_{AdS}(1,2,-3,J=1)$	$=$	-16.255434067000426
$\delta E_{S^5}(1,2,-3,J=1)$	$=$	-20.137332508389193
$\delta E_{\rm Fermi}(1,2,-3,J=1)$	$=$	-18.19638328769481

Table 15: Exact numerical eigenvalues of three-impurity protected sectors

to a numerical diagonalization, we must maintain a high level of precision in the numerical computation. With the parameter choices of (4.12), the numerical diagonalization of the full $2,048\times 2,048$ perturbation matrices on both the spacetime boson (table 3) and spacetime fermion (table 4) sectors yields the spectrum and multiplicities displayed in table 16. The multiplicities are consistent with the superconformal multiplet structure we found in the one-loop analysis (given in table 11). The predicted closed sector eigenvalues (listed in table 15) match, to the precision of the calculation, entries in the list of numerical eigenvalues. These energies also appear at the expected levels within the multiplets. $E_{AdS}(1,2,-3,J)$ and $E_{S^5}(1,2,-3,J)$ appear in bosonic levels with multiplicity 280_B, while energy $E_{\rm Fermi}(1,2,-3,J)$ appears as a fermionic level with multiplicity 560_F; according to table 11 these are uniquely identified as the central $L=4$ levels of their respective multiplets, exactly where the protected energy levels must lie. All of this is clear evidence that the 'closed sector' states of the string theory do not mix with other states under the perturbing Hamiltonian, thus justifying our method of calculating their exact eigenenergies.

At one loop, we found that the three superconformal multiplets were displaced from each other by precisely the internal level spacing. This led to an accidental degeneracy which is lifted in the exact dimension formulas we have just derived. To explore this, it is useful to have formulas for the eigenvalues of all the levels in each multiplet. ¿From the discussion in section 3.3, we see that each level in the string energy spectrum can be connected by a simple integer shift in the angular momentum J. Since we are working at $O(1/J)$ in a large-J expansion, all contributions from this shift must come from the BMN limit of the theory. In other words, by sending $J\to J+2-L/2$ in the BMN formula for the energy

$$E=\sqrt{1+\frac{n^2 g_{YM}^2 N_c}{(J+2-L/2)^2}}+\dots\ , \tag{4.14}$$

we can generate an expansion, to arbitrary order in λ', for each level L in the entire superconformal multiplet.

$\delta E(1,2,-3,J=1)\ \lambda'=1$	Mult.
-30.821354623065	4_B
-26.9394561816763	4_B
-26.2093998737015	64_B
-25.4793435657269	112_B
-21.5974451243382	112_B
-20.8673888163637	448_B
-20.1373325083891	280_B
-16.2554340670003	280_B
-15.5253777590258	448_B
-14.7953214510512	112_B
-10.9134230096624	112_B
-10.1833667016878	64_B
-9.4533103937133	4_B
-5.57141195232456	4_B

$\delta E(1,2,-3,J=1)\ \lambda'=1$	Mult.
-28.8804054023706	8_F
-28.150349094396	32_F
-24.2684506530072	32_F
-23.5383943450326	224_F
-22.808338037058	224_F
-18.9264395956693	224_F
-18.1963832876947	560_F
-17.4663269797201	224_F
-13.5844285383314	224_F
-12.8543722303568	224_F
-12.1243159223822	32_F
-8.24241748099347	32_F
-7.51236117301893	8_F

(4.13)

Table 16: All loop numerical spectrum of three-impurity states ($q=1,\ r=2,\ s=-3,\ \lambda'=1,\ J=1$). Left panel: bosons; right panel: fermions

For the vector $SO(4)_{AdS}$ multiplet, we find

$$\begin{aligned}\delta E_{AdS}(q,r,J,L) \;\approx\; & \frac{1}{J}\Big\{(L-6)(q^2+qr+r^2)\lambda' - \frac{1}{2}(L-4)(q^2+qr+r^2)^2\lambda'^2 \\ & +\frac{3}{16}\Big[2(L-4)q^6+6(L-4)q^5r+5(3L-14)q^4r^2+20(L-5)q^3r^3 \\ & \qquad +5(3L-14)q^2r^4+6(L-4)qr^5+2(L-4)r^6\Big]\lambda'^3 \\ & -\frac{(q^2+qr+r^2)}{16}\Big[5(L-4)q^6+15(L-4)q^5r+(50L-247)q^4r^2+(75L-394)q^3r^3 \\ & \qquad +(50L-247)q^2r^4+15(L-4)qr^5+5(L-4)r^6\Big]\lambda'^4+\ldots\Big\} \qquad (4.15)\end{aligned}$$

(for convenience in eventual comparison with the gauge theory, the eigenvalues have been

expanded to $O(\lambda'^4)$). The corresponding result for the $SO(4)_{S^5}$ vector multiplet is

$$\begin{aligned}\delta E_{S^5}(q,r,J,L) &\approx \frac{1}{J}\Big\{(L-8)(q^2+qr+r^2)\lambda' - \frac{1}{2}(L-6)(q^2+qr+r^2)^2\lambda'^2 \\ &+\frac{3}{16}\Big[2(L-6)q^6+6(L-6)q^5r+(15L-92)q^4r^2+4(5L-31)q^3r^3 \\ &\quad +(15L-92)q^2r^4+6(L-6)qr^5+2(L-6)r^6\Big]\lambda'^3 \\ &-\frac{(q^2+qr+r^2)}{16}\Big[5(L-6)q^6+15(L-6)q^5r+(50L-309)q^4r^2+3(25L-156)q^3r^3 \\ &\quad +(50L-309)q^2r^4+15(L-6)qr^5+5(L-6)r^6\Big]\lambda'^4+\ldots\Big\}\,. \qquad (4.16)\end{aligned}$$

Finally, the result for the spinor multiplet is

$$\begin{aligned}\delta E_{\rm Fermi}(q,r,J,L) &\approx \frac{1}{J}\Big\{(L-7)(q^2+qr+r^2)\lambda' - \frac{1}{2}(L-5)(q^2+qr+r^2)^2\lambda'^2 \\ &+\frac{3}{16}\Big[2(L-5)q^6+6(L-5)q^5r+3(5L-27)q^4r^2+4(5L-28)q^3r^3 \\ &\quad +3(5L-27)q^2r^4+6(L-5)qr^5+2(L-5)r^6\Big]\lambda'^3 \\ &-\frac{(q^2+qr+r^2)}{16}\Big[5(L-5)q^6+15(L-5)q^5r+2(25L-139)q^4r^2+(75L-431)q^3r^3 \\ &\quad +2(25L-139)q^2r^4+15(L-5)qr^5+5(L-5)r^6\Big]\lambda'^4+\ldots\Big\}\,. \qquad (4.17)\end{aligned}$$

It is important to remember that, to obtain the energies of the states as opposed to the energy shifts δE, we must add the BMN energy of the original degenerate multiplet to the above results:

$$\begin{aligned}E_{\rm BMN} &= \sqrt{1+\lambda' q^2}+\sqrt{1+\lambda' r^2}+\sqrt{1+\lambda'(q+r)^2} \\ &= 3+(q^2+r^2+qr)\lambda' - \frac{1}{4}(q^2+r^2+qr)^2\lambda'^2+\ldots \qquad (4.18)\end{aligned}$$

We can conclude from the above formulas that all three multiplets have a common internal level spacing given by the following function of λ' and mode indices:

$$\begin{aligned}\frac{\delta E}{\delta L} &\approx \frac{1}{J}\Big\{(q^2+qr+r^2)\lambda' - \frac{1}{2}\Big[(q^2+qr+r^2)^2\Big]\lambda'^2 \\ &+\frac{3}{16}\Big[2q^6+6q^5r+15q^4r^2+20q^3r^3+15q^2r^4+6qr^5+2r^6\Big]\lambda'^3 \\ &-\frac{(q^2+qr+r^2)}{16}\Big[5q^6+15q^5r+50q^4r^2+75q^3r^3+50q^2r^4+15qr^5+5(r^6\Big]\lambda'^4+\ldots\Big\}\,. \qquad (4.19)\end{aligned}$$

We have expanded in powers of λ', but an all-orders formula can easily be constructed. The multiplets are displaced from one another by shifts that also depend on λ' and mode indices. We note that the one-loop degeneracy between different multiplets (see table 11) is preserved to second order in λ', but is broken explicitly at three loops. At this order and beyond, each multiplet acquires a constant overall (L-independent) shift relative to the other two.

4.2 Two equal mode indices: $(q = r = n,\ s = -2n)$

An independent analysis is required when two mode indices are equal (specifically, we choose $q = r = n,\ s = -2n$). The all-loop matrix elements are complicated and we will refrain from giving explicit expressions for them (though the complete formulas can be found at [20]). As in the unequal mode index case, however, exact eigenvalues can easily be extracted by projection onto certain protected subsectors. In particular, the energy shift for states created by three bosonic mode creation operators with symmetric-traceless $SO(4)_{AdS}$ vector indices (the $\mathfrak{sl}(2)$ sector) turns out to be

$$\begin{aligned}\delta E_{AdS}(n,J) &= -\frac{n^2\lambda'}{J(1+n^2\lambda')\sqrt{4n^2+1/\lambda'}}\left\{\sqrt{4n^2+\frac{1}{\lambda'}}\left(3+4n^2\lambda'\right)+\omega_n\left(4+8n^2\lambda'\right)\right\}\\ &\approx \frac{1}{J}\left\{-7n^2\lambda'+n^4\lambda'^2-\frac{17}{2}n^6\lambda'^3+\dots\right\}. \end{aligned} \tag{4.20}$$

The leading order term in the small-λ' expansion is the $-7/3$ $[280_B]$ level $L=4$ eigenvalue in the Λ_2 multiplet in table 14. The energy shift of the $SO(4)_{S^5}$ partners of these states (belonging to the $\mathfrak{su}(2)$ closed sector) is

$$\begin{aligned}\delta E_{S^5}(n,J) &= -\frac{n^2\lambda'}{J(1+n^2\lambda')\sqrt{4n^2+1/\lambda'}}\left\{\sqrt{4n^2+\frac{1}{\lambda'}}\left(5+4n^2\lambda'\right)+\omega_n\left(6+8n^2\lambda'\right)\right\}\\ &\approx \frac{1}{J}\left\{-11n^2\lambda'+8n^4\lambda'^2-\frac{101}{4}n^6\lambda'^3+\dots\right\}. \end{aligned} \tag{4.21}$$

The one-loop correction corresponds to the $-11/3$ $[280_B]$ level in the Λ_1 submultiplet of table 14. As noted above, the protected symmetrized-fermion ($\mathfrak{su}(2|3)$) sector does not appear when two mode indices are equal. As in the previous section, we can do a numerical diagonalization of the full perturbation matrix to verify that the predicted eigenvalues are indeed exact and closed, but we will omit the details.

By invoking the angular momentum shift $J \to J+2-L/2$ in the BMN limit, we can use the energy shift of the $L=4$ level to recover the exact energy shifts of all other levels in the superconformal multiplets of table 14. The energy shifts of the vector multiplet containing the protected $SO(4)_{AdS}$ bosonic irrep at level $L=4$ are given by the expression

$$\begin{aligned}\delta E_{AdS}(n,J,L) \approx \frac{1}{J}\Big\{&\frac{1}{2}(3L-19)n^2\lambda'-\frac{1}{2}(9L-38)n^4\lambda'^2+\frac{1}{8}(99L-464)n^6\lambda'^3\\ &-\frac{1}{16}(645L-3160)n^8\lambda'^4+\dots\Big\}. \end{aligned} \tag{4.22}$$

The shifts of the multiplet containing the protected $SO(4)_{S^5}$ bosonic irrep are given by

$$\delta E_{S^5}(n,J,L) \approx \frac{1}{J}\Big\{\frac{1}{2}(3L-23)n^2\lambda' - \frac{1}{2}(9L-52)n^4\lambda'^2 + \frac{1}{8}(99L-598)n^6\lambda'^3 - \frac{1}{16}(645L-3962)n^8\lambda'^4 + \dots\Big\}\,. \tag{4.23}$$

Once again, we note that in order to get energies, rather than energy shifts, one must append the BMN energy of the original degenerate multiplet to these results. Unlike the unequal mode index case, there is no accidental degeneracy between superconformal multiplets spanning the three-impurity space, even at one loop in λ'. The level spacings within the two superconformal multiplets are the same, but the multiplets are offset from each other by an L-independent shift (but one that depends on λ' and mode indices).

5 Gauge theory anomalous dimension comparison

In the previous sections, we have given a complete analysis of the perturbed energy spectrum of three-impurity string states. The 'data' are internally consistent in the sense that the perturbed energy levels organize themselves into proper superconformal multiplets of the classical nonlinear sigma model governing the string worldsheet dynamics. Since the quantization procedure leaves only a subgroup of the full symmetry group as a manifest, linearly realized symmetry, this is by itself a nontrivial check on the consistency of the action and quantization procedure. To address the issue of AdS/CFT duality, we must go further and compare the string energy spectrum with the anomalous dimensions of gauge theory operators dual to the three-impurity string states.

The task of finding the anomalous dimensions of BMN operators in the limit of large R-charge and dimension D, but finite $\Delta = D - R$, is greatly simplified by the existence of an equivalence between the dilatation operator of $\mathcal{N} = 4$ SYM and the Hamiltonian of a one-dimensional spin chain. This correspondence was first proposed by Minahan and Zarembo [4], who showed that the anomalous dimension matrix in the $\mathfrak{so}(6)$ subsector of the theory, expanded to one loop in the 't Hooft coupling $\lambda = g_{YM}^2 N_c$, is equivalent to the Hamiltonian of an integrable spin chain. Beisert and Staudacher then showed that a more elaborate integrable spin chain describes the action of the one-loop anomalous dimension operator on the general single-trace operator [18]. (The complete one-loop dilatation operator was derived in [12].) The key to this development is the fact that the one-loop spin chain Hamiltonian has only nearest-neighbor interactions (in the planar large-N_c limit) and is of limited complexity. Higher-loop gauge theory physics is encoded in increasingly long-range spin chain interactions which generate a rapidly growing number of possible terms in the Hamiltonian [5]. Fixing the coefficients of all these terms by comparison with diagrammatic computations would be a very impractical approach. Fortunately, Beisert was able to show that, at least for BMN operators in the $\mathfrak{su}(2)$ closed subsector, general requirements (such as the existence of a well-defined BMN scaling limit) suffice to fix the form of the spin chain

Hamiltonian out to three-loop order [6, 21]. In this section we will discuss the use of these higher-loop spin chains to generate the information we need on the anomalous dimensions of three-impurity operators. We will summarize the salient points here, leaving most of the details to a separate paper [11].

We have already noted that there are three closed subsectors of BMN operators in which impurities taken from a subalgebra of the full superconformal algebra mix only with themselves: we have referred to them as the $\mathfrak{sl}(2)$, $\mathfrak{su}(2)$ (both bosonic) and $\mathfrak{su}(2|3)$ (fermionic) sectors. We will focus our attention on these sectors because their spin chain description is simple and their anomalous dimensions fix the dimensions of the remaining three-impurity operators in the theory. Spin-chain Hamiltonians incorporating higher-loop-order gauge theory physics have been constructed for the $\mathfrak{su}(2)$ and $\mathfrak{su}(2|3)$ sectors but, as far as we know, the $\mathfrak{sl}(2)$ spin chain is known only to one-loop order.

Although these spin chains are integrable, methods such as the Bethe ansatz technique do not immediately yield the desired results for multiple-impurity anomalous dimensions. Minahan and Zarembo did use the Bethe ansatz for the one-loop $\mathfrak{so}(6)$ spin chain (of which the exactly closed $\mathfrak{su}(2)$ system is a subsector) to obtain approximate multi-impurity anomalous dimensions [4], but we need results for all sectors and for higher-loop spin chains. As mentioned above, the $\mathfrak{sl}(2)$ spin chain has phenomenological applications and has been extensively developed in that context. It is therefore possible that some of the results we need can be extracted from the relevant literature.[4] In the end, since we are looking for a unified approach that can handle all sectors and any number of loops, we decided that numerical methods are, at the moment, the most effective way to extract the information we need about gauge theory anomalous dimensions. Since Bethe ansatz equations exist for most of the results that are of interest to us, the numerical results obtained here can eventually be checked against the Bethe-ansatz methodology: these exercises will be reserved for [11].

We begin with a discussion of the bosonic $\mathfrak{sl}(2)$ sector. For total R-charge $\mathfrak{L}$ (the R-charge is equal to the number of lattice sites $\mathfrak{L}$ in this sector), the basis for this system consists of single-trace operators of the form

$$\mathrm{Tr}\left(\mathcal{D}^I Z\ Z^{\mathfrak{L}-1}\right),\ \mathrm{Tr}\left(\mathcal{D}^{I-1} Z\ \mathcal{D}Z\ Z^{\mathfrak{L}-2}\right),\ \mathrm{Tr}\left(\mathcal{D}^{I-1} Z\ Z\mathcal{D}Z\ Z^{\mathfrak{L}-3}\right),\ \ldots\ , \tag{5.1}$$

where Z is the $SO(6)$ Yang-Mills boson carrying one unit of R-charge, $\mathcal{D}$ is a spacetime covariant derivative operator that scales under the chosen $\mathfrak{sl}(2)$ subgroup of the Lorentz group, I is the total impurity number and the full basis contains all possible distributions of $\mathcal{D}$ operators among the Z fields. Conservation of various $U(1)$ subgroups of the R-symmetry group ensures that operators of this type mix only among themselves to all orders in the gauge theory (as long as we work in the 't Hooft large-N_c limit). This gauge theory closed subsector corresponds to the symmetric traceless irrep of $SO(4)_{AdS}$ bosons in the string theory (states whose energy shifts are given in eqns. (4.9) and (4.20)).

The one-loop spin chain Hamiltonian for this sector has been derived by Beisert [12] in a representation where there is a lattice site assigned to each Z field, and each site supports a

[4]We thank A. Belitsky for making us aware of this literature and for helpful discussions on this point.

harmonic oscillator whose level of excitation counts the number of $\mathcal{D}$ operators acting on that Z insertion. The raising operator $a_i^\dagger$ therefore corresponds to the insertion of a derivative at the i^{th} lattice site:

$$|\mathfrak{L}\rangle \sim \mathrm{Tr}\left(Z^{\mathfrak{L}}\right) \ , \qquad (a_i^\dagger)^n\,|\mathfrak{L}\rangle \sim \mathrm{Tr}\left(Z^{i-1}\mathcal{D}^n Z Z^{\mathfrak{L}-i}\right) \ , \ \ldots \tag{5.2}$$

The spin chain Hamiltonian is a sum over $\mathfrak{L}$ lattice sites of a nearest-neighbor interaction which moves excitations between neighbors while keeping the net excitation number (number of impurities I) fixed:

$$\begin{aligned} H^{\mathfrak{sl}(2)} &= \frac{\lambda}{8\pi^2}\sum_{k=1}^{\mathfrak{L}} H^{\mathfrak{sl}(2)}_{k,k+1} \ , \\ H^{\mathfrak{sl}(2)}_{1,2}(a_1^\dagger)^k(a_2^\dagger)^{n-k}\,|\mathfrak{L}\rangle &= \sum_{k'=0}^{n}\left[\delta_{k=k'}\left(h(k)+h(n-k)\right) - \frac{\delta_{k\neq k'}}{|k-k'|}\right](a_1^\dagger)^{k'}(a_2^\dagger)^{n-k'}\,|\mathfrak{L}\rangle \ . \end{aligned} \tag{5.3}$$

The weighting of different terms by the harmonic numbers $h(n) = \sum_{r=1}^{n} r^{-1}$ is a general feature of spin chains derived from gauge theories and reflects the infrared behavior of gluon emission.[5] The impurity number I is the net excitation number of the $\mathfrak{L}$ oscillators. The interaction propagates impurities around the lattice and assigns special amplitudes to 'collisions' of multiple excitations on a single site. Our interest is in finding the spectrum of energies of three impurities on a large lattice under these dynamics.

When $\mathfrak{L}$ is large and the number of impurities is fixed, it is natural to seek a virial expansion for the eigenvalues of $H^{\mathfrak{sl}(2)}$. The above Hamiltonian acts on an isolated impurity exactly like the usual lattice Laplacian; it can be diagonalized by passing to momentum space. The rest of the Hamiltonian amounts to two- and higher-body scattering vertices for the single-impurity pseudoparticles, and standard many-body intuition tells us that such interaction vertices are suppressed by powers of $\mathfrak{L}^{-1}$ compared to the leading pseudoparticle energies. Since we are in fact looking for an expansion of the energy eigenvalues in powers of $\mathfrak{L}^{-1}$, it is useful to rewrite $H^{\mathfrak{sl}(2)}$ in terms of momentum space creation and annihilation operators $a_p,\ a_p^\dagger$ defined by the usual discrete Fourier transform. Since the definition of $H^{\mathfrak{sl}(2)}$ is given in terms of its action on states, rather than as an explicit operator, this rewriting of the Hamiltonian takes a little work, but the result is simple [22]:

$$\begin{aligned} H^{\mathfrak{sl}(2)} = \frac{\lambda}{8\pi^2}\Big[&\sum_p 4\sin^2\frac{p\pi}{\mathfrak{L}}\ a_p^\dagger a_p \\ &+\frac{1}{\mathfrak{L}}\sum_{p,q,r,s}\delta_{p+q,r+s}\left(-\sin^2\frac{p\pi}{\mathfrak{L}} - \sin^2\frac{q\pi}{\mathfrak{L}} + \sin^2\frac{(p+q)\pi}{\mathfrak{L}}\right) a_p^\dagger a_q^\dagger a_r a_s\ \Big] + \ldots \end{aligned} \tag{5.4}$$

The pseudoparticle creation and annihilation operators $a_p,\ a_p^\dagger$ are labeled by integer momenta $p = 0, 1, \ldots, \mathfrak{L}-1$ and obey the standard algebra. The ellipses stand for three- and higher-body interactions, which are suppressed by even higher powers of $\mathfrak{L}^{-1}$. For I impurities

[5] We are indebted to A. Belitsky for clarifying discussions on this point.

carrying quantized pseudoparticle momenta n_i, this leads to an energy formula

$$E_{\mathfrak{L}} = I + \frac{\lambda}{2\pi^2}\sum_{i=1}^{I}\sin^2\frac{n_i\pi}{\mathfrak{L}} + \frac{\lambda}{\mathfrak{L}^3}V_{\rm two-body}(n_1,\ldots,n_I) + \ldots \tag{5.5}$$

To facilitate comparison with string theory, we have reinstated the zeroth-order term for the total dimension minus R-charge of an I-impurity operator. The true eigenvalues differ by small corrections from the lattice Laplacian energies of the free pseudoparticles, which are labeled by the pseudoparticle momenta n_i. These integers are the gauge theory analogs of the string mode indices and, to make contact with string theory, we must take the limit $\mathfrak{L}\to\infty$ while keeping the n_i fixed. In this limit, the eigenvalues of eqn. (5.5) will scale as

$$E_{\mathfrak{L}}(\{n_i\}) = I + \frac{\lambda}{\mathfrak{L}^2}E^{(1,2)}(\{n_i\}) + \frac{\lambda}{\mathfrak{L}^3}E^{(1,3)}(\{n_i\}) + O(\lambda\mathfrak{L}^{-4})\ . \tag{5.6}$$

This is the leading-order (in λ) version of the more general BMN scaling

$$E(\lambda,J) \simeq E(\lambda/J^2,1/J) = \sum_{i=1,\ j=0} E^{(i,2i+j)}\left(\frac{\lambda}{J^2}\right)^i\ J^{-j}\ , \tag{5.7}$$

which, as we have seen in previous sections, naturally characterizes string energy levels. In the case at hand, where we are computing energies to $O(\lambda)$ only, BMN scaling is a consequence of the $\mathfrak{L}^{-2}$ scaling of energy eigenvalues that follows automatically from the form of the virial Hamiltonian (5.4). Whether the spectra of higher-order spin chain Hamiltonians scale with $\mathfrak{L}$ in accordance with eqn. (5.7) is a very nontrivial question which we will address shortly.

To compare with the corresponding string theory predictions of eqns. (4.9) and (4.20), we reorganize those results as follows: we reinstate the BMN energy of the degenerate multiplet (4.18) (expanded to first order in λ'); we replace λ' with λ/J^2 and replace J by $\mathfrak{L}$. This gives specific string theory predictions for the large-$\mathfrak{L}$ scaling of one-loop anomalous dimensions of the AdS, or $\mathfrak{sl}(2)$, closed sector. As usual, there are two distinct cases: for unequal mode indices ($q\neq r\neq s=-q-r$), we have

$$E_{AdS}(q,r,\mathfrak{L}) = 3 + (\mathfrak{L}-2)(q^2+r^2+qr)\frac{\lambda}{\mathfrak{L}^3} + O(\mathfrak{L}^{-4})\ , \tag{5.8}$$

while for pairwise equal mode indices $(n,n,-2n)$ we have

$$E_{AdS}(n,\mathfrak{L}) = 3 + (3\mathfrak{L}-7)n^2\frac{\lambda}{\mathfrak{L}^3} + O(\mathfrak{L}^{-4})\ . \tag{5.9}$$

This matches the expected virial scaling of the spin chain eigenvalues displayed in eqn. (5.6), with the specific identifications

$$E^{(1,2)}_{AdS} = (q^2+r^2+qr) \qquad E^{(1,3)}_{AdS} = -2(q^2+r^2+qr) \qquad E^{(1,3)}_{AdS}/E^{(1,2)}_{AdS} = -2 \tag{5.10}$$

for $q \neq r \neq s = -q - r$, or

$$E^{(1,2)}_{AdS} = 3n^2 \qquad E^{(1,3)}_{AdS} = -7n^2 \qquad E^{(1,3)}_{AdS}/E^{(1,2)}_{AdS} = -7/3 \tag{5.11}$$

for $q = r = n$ and $s = -2n$.

To check these predictions, we numerically diagonalize the spin chain for three impurities for a sequence of values of $\mathfrak{L}$ (up to $30 \lesssim \mathfrak{L} \lesssim 60$ in practice). It is a matter of convenience whether we construct the $\mathfrak{L} \times \mathfrak{L}$ Hamiltonian matrix using the position space version (5.3) or the momentum space version (5.4) of the Hamiltonian. We then track how the eigenvalues evolve as $\mathfrak{L}$ varies and fit the data to a general $\mathfrak{L}^{-1}$ expansion in order to extract the spin chain coefficients $E^{(1,2)}_{\mathfrak{sl}(2)}$, $E^{(1,3)}_{\mathfrak{sl}(2)}$ (the vanishing of the coefficient $E^{(1,1)}_{\mathfrak{sl}(2)}$ is a check of BMN scaling, but is essentially guaranteed here). The results of this exercise, presented in table 17, show clear agreement between the string and gauge theory predictions. In the fourth column we list the ladder of string mode indices that correspond to the succession of eigenstates in the gauge theory, according to eqns. (5.10) and (5.11). For the low-lying states in the spectrum, the numerical agreement is convincing. As one moves up the ladder of energies, higher-order $1/\mathfrak{L}$ corrections become more important. We would have to generate data up to higher lattice sizes and do a more precise scaling fit to improve the agreement at higher levels in the spectrum.

At this point it is appropriate to say a few words about the role of integrability in this problem. It was first argued in [23] that the complete GS action of IIB superstring theory on $AdS_5 \times S^5$ is integrable. Integrability has since taken a central role in studies of the AdS/CFT correspondence, as any precise non-perturbative understanding of integrability on both sides of the duality would be extremely powerful. Integrability on either side of the duality gives rise to an infinite tower of hidden charges that can be loosely classified as either local (Abelian) or non-local (non-Abelian). In the Abelian sector, contact between the integrable structures of gauge theory and semiclassical string theory (a subject which was first investigated in [24]) has been made to two loops in λ (see, eg., [10, 25, 26, 27]). (The corresponding problem in the non-local sector was addressed to one-loop order in [28, 29].) One of the local gauge theory charges, denoted by Q_2, can be shown to anticommute in the $\mathfrak{su}(2)$ sector with a parity operator P (to three loops in λ), whose action on a single-trace state in the gauge theory is to invert the order of all fields within the trace [6, 5]. Furthermore, Q_2 can be shown to connect states of opposite parity. Taken together with the conservation of Q_2, these facts imply that all eigenstates in the spectrum connected by P must be degenerate. These degenerate states are known as parity pairs and their existence can be interpreted as a necessary (but not sufficient) condition for integrability. The spectrum in table 17 exhibits such a degeneracy and makes it clear that parity pairs are simply distinct states whose lattice momenta (or worldsheet mode indices) are related by an overall sign flip. Since the net momentum of allowed states is zero, parity pair states can in principle scatter into each other, and their degeneracy is a non-trivial constraint on the interactions. As a small caveat, we note that lattice momentum conservation implies that mixing of parity-pair states can only occur via connected three-body (or higher) interactions. As the virial analysis shows, at the order to which we are working, only two-body interactions

$E^{(1,2)}_{\mathfrak{sl}(2)}$	$E^{(1,3)}_{\mathfrak{sl}(2)}$	$E^{(1,3)}_{\mathfrak{sl}(2)}/E^{(1,2)}_{\mathfrak{sl}(2)}$	String Modes (q,r,s)	% Error
$1+1.2\times10^{-9}$	$-2-3.1\times10^{-7}$	$-2-3.1\times10^{-7}$	$(1,0,-1)$	0.00002%
$3-7.6\times10^{-9}$	$-7+1.9\times10^{-6}$	$-7/3+6.3\times10^{-7}$	$(1,1,-2)$	0.00001%
$3-7.6\times10^{-9}$	$-7+1.9\times10^{-6}$	$-7/3+6.3\times10^{-7}$	$(-1,-1,2)$	0.00001%
$4-2.8\times10^{-7}$	$-8+6.9\times10^{-6}$	$-2+1.7\times10^{-6}$	$(2,0,-2)$	0.0001%
$7-2.9\times10^{-7}$	$-14+7.1\times10^{-5}$	$-2+1.0\times10^{-5}$	$(1,2,-3)$	0.0005%
$7-2.9\times10^{-7}$	$-14+7.1\times10^{-5}$	$-2+1.0\times10^{-5}$	$(-1,-2,3)$	0.0005%
$9-4.1\times10^{-7}$	$-18+1.0\times10^{-4}$	$-2+1.0\times10^{-5}$	$(3,0,-3)$	0.0005%
$12+8.4\times10^{-7}$	$-28-1.5\times10^{-4}$	$-7/3-1.2\times10^{-5}$	$(2,2,-4)$	0.0003%
$12+8.4\times10^{-7}$	$-28-1.5\times10^{-4}$	$-7/3-1.2\times10^{-5}$	$(-2,-2,4)$	0.0003%
$13-7.0\times10^{-6}$	$-26+1.7\times10^{-3}$	$-2+1.3\times10^{-4}$	$(1,3,-4)$	0.01%
$13-7.0\times10^{-6}$	$-26+1.7\times10^{-3}$	$-2+1.3\times10^{-4}$	$(-1,-3,4)$	0.01%
$16-1.4\times10^{-6}$	$-32+3.9\times10^{-4}$	$-2+2.4\times10^{-5}$	$(4,0,-4)$	0.002%
$19-7.5\times10^{-6}$	$-38+2.2\times10^{-3}$	$-2+1.1\times10^{-4}$	$(2,3,-5)$	0.01%
$19-7.5\times10^{-6}$	$-38+2.2\times10^{-3}$	$-2+1.1\times10^{-4}$	$(-2,-3,5)$	0.01%
$21-3.4\times10^{-6}$	$-42+8.8\times10^{-4}$	$-2+4.2\times10^{-5}$	$(1,4,-5)$	0.002%
$21-3.4\times10^{-6}$	$-42+8.8\times10^{-4}$	$-2+4.2\times10^{-5}$	$(-1,-4,5)$	0.002%

Table 17: Scaling limit of numerical spectrum of three-impurity $\mathfrak{sl}(2)$ sector at one-loop order

are present and the parity pair degeneracy is automatic. The same remark applies to the string theory analysis to $O(J^{-1})$ in the curvature expansion. A calculation of the string theory spectrum carried out to $O(J^{-2})$ is needed to see whether parity pair degeneracy survives string worldsheet interactions; a discussion of this point will be given in [30].

We now turn to the closed $\mathfrak{su}(2)$ sector of gauge theory operators, corresponding to the symmetric-traceless bosonic $SO(4)_{S^5}$ sector of the string theory. The operator basis for this sector consists of single-trace monomials built out of two complex scalar fields Z and ϕ, where Z is the complex scalar carrying one unit of charge under the $U(1)$ R-charge subgroup and ϕ is one of the two scalars with zero R-charge, transforming as an $SO(4)$ vector in the $SO(6)\simeq U(1)_R\times SO(4)$ decomposition of the full R-symmetry group of the gauge theory. The collection of operators

$$\mathrm{tr}(\phi^I Z^{\mathfrak{L}-I}),\ \mathrm{tr}(\phi^{I-1}Z\phi Z^{\mathfrak{L}-I-1}),\ \mathrm{tr}(\phi^{I-2}Z\phi^2 Z^{\mathfrak{L}-I-1}),\ \ldots \tag{5.12}$$

(and all possible permutations, modulo cyclic equivalence, of the $\mathfrak{L}$ factors) forms a basis with I impurities and R-charge equal to $\mathfrak{L}-I$. The anomalous dimension operator simply permutes these monomials among themselves in ways that get more elaborate as we go to higher loop orders in the gauge theory. An explicit spin-chain Hamiltonian which incorporates gauge theory physics up to three loops has been constructed by enumerating all

possible interaction terms and fixing coefficients by demanding BMN scaling behavior of the spectrum for large lattice size [6, 21, 5]. We now turn to a numerical analysis of scaling in this sector in order to examine the match to string theory predictions at higher loop orders.

The complete Hamiltonian in this sector will be written as a sum of terms of increasing order in the coupling constant λ:

$$H^{\mathfrak{su}(2)} = \sum_n \left(\frac{\lambda}{8\pi^2}\right)^n H_{2n}^{\mathfrak{su}(2)} \ . \tag{5.13}$$

The action of the different terms $H_{2n}^{\mathfrak{su}(2)}$ on the operator basis of eqn. (5.12) can be built out of permutation operators P_{ij} which exchange the fields on the i^{th} and j^{th} sites on a lattice of $\mathfrak{L}$ sites. Using the compact notation

$$\{n_1, n_2, \dots\} = \sum_{k=1}^{L} P_{k+n_1,\ k+n_1+1} P_{k+n_2,\ k+n_2+1} \cdots \ , \tag{5.14}$$

Beisert, Kristjansen and Staudacher [5] find the following explicit forms for the one- and two-loop terms in the Hamiltonian (we will return to the question of three-loop terms shortly):

$$H_2^{\mathfrak{su}(2)} = 2\left(\{\} - \{0\}\right) \ , \qquad H_4^{\mathfrak{su}(2)} = 2\big(-4\{\} + 6\{0\} - (\{0,1\} + \{1,0\})\big) \ . \tag{5.15}$$

Just as in the $\mathfrak{sl}(2)$ case, it is clear that for fixed impurity number I and large lattice size $\mathfrak{L}$, there is a virial expansion of the one-loop energy eigenvalues essentially identical to eqn. (5.5). We therefore expect the exact energy eigenvalues to be labeled by integer pseudoparticle momenta.

To obtain the dependence of the spectrum on λ, our strategy will be to develop a standard Rayleigh-Schrödinger perturbation theory expansion, treating $H_2^{\mathfrak{su}(2)}$ as a zeroth-order Hamiltonian (obtaining its eigenvalues and eigenvectors numerically), then using non-degenerate perturbation theory in $H_4^{\mathfrak{su}(2)}$ to obtain the next-order corrections (i.e. taking expectation values of $H_4^{\mathfrak{su}(2)}$ in the eigenvectors of $H_2^{\mathfrak{su}(2)}$). The expansion coefficients of each eigenvalue are numbers which depend on the lattice size $\mathfrak{L}$ in some non-explicit way: we have to do the calculation for many values of $\mathfrak{L}$ and perform an extrapolation in $\mathfrak{L}^{-1}$ in order to find the information of interest to us but, as we will see, this is not too difficult. The only possible obstruction to this program would be a degeneracy in the spectrum of $H_2^{\mathfrak{su}(2)}$ which would oblige us to use degenerate perturbation theory. Although the spectrum of $H_2^{\mathfrak{su}(2)}$ is indeed degenerate, the higher charge Q_2 constrains matrix elements of the perturbing Hamiltonian $H_4^{\mathfrak{su}(2)}$ in such a way that a) the first-order perturbation theory calculation can proceed as if the spectrum were non-degenerate and b) the degeneracy is maintained to this order.

These considerations lead us to a numerical scheme involving a series of steps. First, we find the eigenvalues and eigenvectors of $H_2^{\mathfrak{su}(2)}$ for the three-impurity Hamiltonian on lattices of length up to $\mathfrak{L} \approx 30$ and fit a power series in $\mathfrak{L}^{-1}$ to the eigenvalues. Table 19 displays the values of the coefficients $E_{\mathfrak{su}(2)}^{(1,2)}, E_{\mathfrak{su}(2)}^{(1,3)}$ (as defined in eqn. 5.6) for the low-lying levels

that we infer from this fit. Second, we obtain a series of values for the $O(\lambda^2)$ corrections to the eigenvalues by taking the expectation value of the perturbing Hamiltonian $H_4^{\mathfrak{su}(2)}$ between the numerical eigenvectors obtained in the previous step. We fit a power series in $\mathfrak{L}^{-1}$ to this data to read off the expansion coefficients $E^{(2,n)}_{\mathfrak{su}(2)}$ for the low-lying levels, with the results displayed in table 20. It is important that the $O(\lambda^2)$ data scales as $\mathfrak{L}^{-4}$ (i.e. that the coefficients of lower powers of $\mathfrak{L}^{-1}$ vanish to numerical accuracy) as required by BMN scaling (see eqn. (5.7)). Since the Hamiltonian was determined in part by requiring this scaling, this is perhaps not a surprise: the real test will be the match of the BMN scaling coefficients to string theory data.

To compare with string theory results for the bosonic symmetric-traceless $SO(4)_{S^5}$ sector eigenvalues, we need to recast eqns. (4.10) and (4.21) as expansions in powers of λ and $\mathfrak{L}^{-1}$. We denote by $E^{(n,m)}_{S^5}$ the coefficient of $\lambda^n \mathfrak{L}^{-m}$ in the large-$\mathfrak{L}$ expansion of the string theory energies: they can be directly compared with the corresponding quantities extracted from the numerical spin chain analysis. The string theory predictions for scaling coefficients, up to second order in λ, are given in table 18. As usual, the predictions for three-impurity states

$E^{(n,m)}_{S^5}$	$(q \neq r \neq s)$	$(q = r = n)$
$E^{(1,2)}_{S^5}$	$(q^2 + qr + r^2)$	$3n^2$
$E^{(1,3)}_{S^5}$	$2(q^2 + qr + r^2)$	$7n^2$
$E^{(2,4)}_{S^5}$	$-\frac{1}{4}(q^2 + qr + r^2)^2$	$-\frac{9}{4}n^4$
$E^{(2,5)}_{S^5}$	$-2(q^2 + qr + r^2)^2$	$-19n^4$

Table 18: String predictions for $\mathfrak{su}(2)$ scaling coefficients, to two loops

with unequal mode indices have to be stated separately from those for states with two equal mode indices. The level of agreement between string theory predictions and the spin chain numerical results is stated in the last column of tables 19 and 20. The fit is very good for low-lying levels and gets worse as we go up in the spectrum. We are confident that, at any given level, the fit may be made arbitrarily precise by generating data out to large enough lattice size. We take these results as strong evidence that the string theory analysis agrees with the gauge theory up to $O(\lambda^2)$ in this sector.

We now turn to a discussion of gauge theory physics beyond two loops. As it happens, the three-loop Hamiltonian can be fixed up to two unknown coefficients (α_1 and α_2) by basic field theory considerations [21]:

$$\begin{aligned} H_6^{\mathfrak{so}(6)} &= (60 + 6\alpha_1 - 56\alpha_2)\,\{\} + (-104 + 14\alpha_1 + 96\alpha_2)\,\{0\} \\ &\quad + (24 + 2\alpha_1 - 24\alpha_2)\,(\{0,1\} + \{1,0\}) + (4 + 6\alpha_1)\,\{0,2\} \\ &\quad (-4 + 4\alpha_2)\,(\{0,1,2\} + \{2,1,0\}) - \alpha_1\,(\{0,2,1\} + \{1,0,2\})\ . \end{aligned} \tag{5.16}$$

Originally, these coefficients were determined by demanding proper BMN scaling in the theory and that the dynamics be integrable at three loops (by requiring that the Hamiltonian

$E^{(1,2)}_{\mathfrak{su}(2)}$	$E^{(1,3)}_{\mathfrak{su}(2)}$	$E^{(1,3)}_{\mathfrak{su}(2)}/E^{(1,2)}_{\mathfrak{su}(2)}$	String Modes (q,r,s)	% Error
$1+1.1\times10^{-9}$	$2-1.6\times10^{-7}$	$2-1.7\times10^{-7}$	$(1,0,-1)$	0.00001%
$3-9.9\times10^{-9}$	$7+1.5\times10^{-6}$	$7/3+5.2\times10^{-7}$	$(1,1,-2)$	0.00002%
$3-1.2\times10^{-8}$	$7+1.8\times10^{-6}$	$7/3+6.2\times10^{-7}$	$(-1,-1,2)$	0.00003%
$4-9.7\times10^{-8}$	$8+1.5\times10^{-5}$	$2+3.8\times10^{-6}$	$(2,0,-2)$	0.0002%
$7-1.6\times10^{-6}$	$14+2.6\times10^{-4}$	$2+3.8\times10^{-5}$	$(1,2,-3)$	0.002%
$7-1.7\times10^{-6}$	$14+2.6\times10^{-4}$	$2+3.8\times10^{-5}$	$(-1,-2,3)$	0.002%
$9-2.7\times10^{-6}$	$18+4.9\times10^{-4}$	$2+5.5\times10^{-5}$	$(3,0,-3)$	0.003%
$12+3.1\times10^{-3}$	27.5	2.29	$(2,2,-4)$	2%
$12-1.1\times10^{-2}$	29.9	2.50	$(-2,-2,4)$	7%

Table 19: Scaling limit of one-loop numerical spectrum of three-impurity $\mathfrak{su}(2)$ subsector

$E^{(2,4)}_{\mathfrak{su}(2)}$	$E^{(2,5)}_{\mathfrak{su}(2)}$	$E^{(2,5)}_{\mathfrak{su}(2)}/E^{(2,4)}_{\mathfrak{su}(2)}$	String Modes (q,r,s)	% Error
-0.24999997	-2.000004	8.00002	$(1,0,-1)$	0.0003%
-2.24999	-19.002	8.445	$(1,1,-2)$	0.01%
-2.24999	-19.002	8.445	$(-1,-1,2)$	0.01%
-4.00003	-31.997	7.9991	$(2,0,-2)$	0.01%
-12.25	-97.99	7.999	$(1,2,-3)$	0.02%
-12.25	-97.99	7.999	$(-1,-2,3)$	0.02%
-20.25	-161.8	7.990	$(3,0,-3)$	0.1%
-36.10	-289.6	8.02	$(2,2,-4)$	5.0%
-36.18	-276.4	7.64	$(-2,-2,4)$	9.5%

Table 20: Scaling limit of two-loop numerical spectrum of three-impurity $\mathfrak{su}(2)$ subsector

H commute with Q_2 up to three loops, and that Q_2 anticommute with P); these assumptions set $\alpha_{1,2}=0$. By studying an $\mathfrak{su}(2|3)$ spin chain model, Beisert [6] was subsequently able to show that independent symmetry arguments, along with BMN scaling, uniquely set $\alpha_1=\alpha_2=0$ (thus proving integrability at three loops).

The three-loop Hamiltonian $H_6^{\mathfrak{su}(2)}$ can be treated as a second-order correction to $H_2^{\mathfrak{su}(2)}$. This allows us to numerically evaluate the $O(\lambda^3)$ contribution to the spectrum by using second-order Rayleigh-Schrödinger perturbation theory (there is an intermediate state sum involved, but since we are doing the calculation numerically, this is not a serious problem). There is also the issue of degeneracy but the existence of a higher conserved charge once again renders the problem effectively non-degenerate (the details of this argument will be

given in a more detailed study of numerical approaches to the spin chain problem [11]). The resulting three-loop data for large-$\mathfrak{L}$ can be fit to a power series in $\mathfrak{L}^{-1}$ to read off the expansion coefficients $E^{3,n}_{\mathfrak{su}(2)}$. It turns out that, to numerical precision, the coefficients are non-vanishing only for $n > 5$ (as required by BMN scaling). The first two non-vanishing coefficients are displayed in table 21 for low-lying levels.

These results can be compared with string theory predictions derived (in the manner described in previous paragraphs) from eqn. (4.10), and the accuracy of the match is displayed in the last column of table 21. The important point is that there is substantial disagreement with string results at $O(\lambda^3)$ for all energy levels: the low-lying states exhibit a mismatch ranging from roughly 18% to 30%, and there is no evidence that this can be repaired by taking data on a larger range of lattice sizes. There is apparently a general breakdown of the correspondence between string theory and gauge theory anomalous dimensions at three loops, despite the precise and impressive agreement at first and second order. This disagreement was first demonstrated in the two-impurity regime [8], and additional evidence was presented more recently in the context of a semiclassical string analysis [31]. It is therefore perhaps not surprising that the three-loop disagreement is reproduced in the three-impurity regime, but it provides us with more information which may help to clarify this puzzling phenomenon.

$E^{(3,7)}_{\mathfrak{su}(2)}/E^{(3,6)}_{\mathfrak{su}(2)}$	String Modes (q,r,s)	%Error
16.004	$(1,0,-1)$	33.4%
14.114	$(1,1,-2)$	18.8%
14.114	$(-1,-1,2)$	18.8%
16.037	$(2,0,-2)$	33.6%
14.272	$(1,2,-3)$	21.7%
14.272	$(-1,-2,3)$	21.7%
15.561	$(3,0,-3)$	29.7%

Table 21: Three-loop numerical spectrum of three-impurity $\mathfrak{su}(2)$ subsector

The same exercise can be repeated for the closed $\mathfrak{su}(2|3)$ fermionic sector, whose string theory dual is comprised of pure fermionic states symmetrized in $SO(4) \times SO(4)$ indices in either the $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ or $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ irreps (projected onto $\Pi_\pm$ subspaces). The spin chain system is embedded in Beisert's $\mathfrak{su}(2|3)$ model, where the fermionic sector of the Hamiltonian has been recorded up to two-loop order [6]. Since the relevant points of the numerical gauge/string comparison have already been made, we will relegate the details of the gauge theory side to [11] and simply state the final results. We also note that Beisert has provided us with the fermionic part of the *three-loop* vertex in the $\mathfrak{su}(2|3)$ sector. The large-$\mathfrak{L}$ spectrum of the three-loop contribution will be scrutinized in [11], but, based on the existing evidence at three-loop order, we do not expect an agreement with string theory.

In this sector, the R-charge and the lattice length are related by $J = \mathfrak{L} - I/2$. The fermionic one- and two-loop string predictions are therefore found from eqn. (4.11) to be

$$E^{(1,2)}_{\text{Fermi}} = (q^2 + qr + r^2) \qquad E^{(1,3)}_{\text{Fermi}} = 0$$
$$E^{(2,4)}_{\text{Fermi}} = -\frac{1}{4}(q^2 + qr + r^2)^2 \qquad E^{(2,5)}_{\text{Fermi}} = -(q^2 + qr + r^2)^2 \ . \tag{5.17}$$

As noted above, this sector does not admit states with equivalent mode indices.

The large-$\mathfrak{L}$ $\mathfrak{su}(2|3)$ fermionic spin chain extrapolations are given at one-loop order in table 22. The two-loop data are obtained using the same first-order perturbation theory treatment described above in the $\mathfrak{su}(2)$ sector; the results are recorded in table 23. The two-loop spectrum is subject to stronger $\mathfrak{L}^{-1}$ corrections, but the data are still convincing and could be improved by running the extrapolation out to larger lattice sizes. The close agreement for the low-lying levels corroborates the match between gauge and string theory up to two-loop order.

$E^{(1,2)}_{\mathfrak{su}(2\|3)}$	$E^{(1,3)}_{\mathfrak{su}(2\|3)}$	$E^{(1,3)}_{\mathfrak{su}(2\|3)}/E^{(1,2)}_{\mathfrak{su}(2\|3)}$	String Modes (q,r,s)	% Error
$1 + 1.3 \times 10^{-10}$	-1.9×10^{-8}	-1.9×10^{-8}	$(1,0,-1)$	0.000002%
$4 - 1.0 \times 10^{-7}$	1.8×10^{-5}	4.6×10^{-6}	$(2,0,-2)$	0.0005%
$7 - 2.5 \times 10^{-7}$	4.4×10^{-5}	6.3×10^{-6}	$(1,2,-3)$	0.0006%
$7 - 2.5 \times 10^{-7}$	4.4×10^{-5}	6.3×10^{-6}	$(-1,-2,3)$	0.0006%
$9 - 3.9 \times 10^{-7}$	7.9×10^{-5}	8.7×10^{-6}	$(3,0,-3)$	0.0009%
$13 - 4.0 \times 10^{-6}$	8.2×10^{-4}	6.3×10^{-5}	$(1,3,-4)$	0.006%
$13 - 4.0 \times 10^{-6}$	8.2×10^{-4}	6.3×10^{-5}	$(-1,-3,4)$	0.006%
$16 - 2.0 \times 10^{-5}$	4.1×10^{-3}	2.6×10^{-4}	$(4,0,-4)$	0.003%
$19 - 3.5 \times 10^{-5}$	7.3×10^{-3}	3.8×10^{-4}	$(2,3,-5)$	0.004%
$19 - 3.5 \times 10^{-5}$	7.3×10^{-3}	3.8×10^{-4}	$(-2,-3,5)$	0.004%

Table 22: Scaling limit of one-loop numerical spectrum of three-impurity $\mathfrak{su}(2|3)$ fermionic subsector

$E^{(2,4)}_{\mathfrak{su}(2\|3)}$	$E^{(2,5)}_{\mathfrak{su}(2\|3)}$	$E^{(2,5)}_{\mathfrak{su}(2\|3)}/E^{(2,4)}_{\mathfrak{su}(2\|3)}$	String Modes (q,r,s)	% Error
-0.25	-0.99999	3.99995	$(1,0,-1)$	0.001%
-4.00006	-15.990	3.998	$(2,0,-2)$	0.06%
-12.251	-48.899	3.992	$(1,2,-3)$	0.2%
-12.251	-48.899	3.992	$(-1,-2,3)$	0.2%
-20.25	-80.89	3.995	$(3,0,-3)$	0.1%
-42.25	-168.2	3.98	$(1,3,-4)$	0.5%
-42.25	-168.2	3.98	$(-1,-3,4)$	0.5%
-64.00	-254.6	3.98	$(4,0,-4)$	0.6%
-90.26	-359.3	3.98	$(2,3,-5)$	0.5%
-90.26	-359.8	3.99	$(-2,-3,5)$	0.3%

Table 23: Scaling limit of two-loop numerical spectrum of three-impurity $\mathfrak{su}(2|3)$ fermionic subsector

6 Conclusions

The BMN/pp-wave mechanism has emerged as a useful proving ground for the postulates of the AdS/CFT correspondence. When the full Penrose limit is lifted, a rich landscape emerges, even in the two-impurity regime, upon which the string and gauge theory sides of the duality have exhibited an intricate and impressive match to two loops in the gauge coupling and first nontrivial order in the curvature expansion. While the conditions under which agreement is obtained are substantially more demanding in the higher-impurity problem, we have shown that this agreement is maintained for three-impurity string states and SYM operators. We expect that these conclusions will persist for four or more impurities. Since the Bethe ansatz results of Minahan and Zarembo [4] provide an all-impurity prediction in the gauge theory, the methods presented here can easily be employed to perform a simple check of this statement, for example, in the closed $SO(4)_{S^5}$ sector of the string theory for four (or higher) impurity string states.

Although the two-loop agreement survives at the three-impurity level, we have also confirmed the previously observed mismatch at three loops in the gauge theory coupling. In [31] it was suggested that this disagreement may be attributed to an order-of-limits problem. This notion was made more precise in [19], where it was noted that a certain class of long-range spin chain interactions may not survive the small-λ expansion in the gauge theory. A thorough, quantitative understanding of this proposal has yet to be obtained, however. In the end, the analyses carried out here will provide an extremely stringent test of any proposed solution to this vexing problem.

Acknowledgements

We would like to thank John Schwarz and Jonathan Heckman for helpful discussions. We also thank Gleb Arutjunov, Niklas Beisert, Andrei Belitsky, Charlotte Kristjansen, Didina Serban and Matthias Staudacher for many helpful comments and suggestions. This work was supported in part by US Department of Energy grants DE-FG02-91ER40671 (Princeton) and DE-FG03-92-ER40701 (Caltech).

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